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TABLE OF CONTENTS

History 1049
Foundations 1051
Algebra 1053
Abstract algebra 1056
Theory of groups 1058
Number theory 1062
Analysis 1067
Theory of sets, theory of functions of real variables 1068
Theory of functions of complex variables 1073
Theory of series 1077
Fourier series and generalizations, integral transforms 1080
Polynomials, polynomial approximations 1082
Special functions 1083
Harmonic functions, potential theory . . 1084
Differential equations 1085
Integral equations 1092
Functional analysis 1092
Theory of probability 1097
Mathematical statistics 1102
Topology 1106
Geometry 1114
Convex domains, extremal problems, integral geometry 1114
Algebraic geometry 1116
Differential geometry 1118
Numerical and graphical methods . . . 1123
Astronomy 1131
Relativity 1132
Mechanics 1135
Hydrodynamics, aerodynamics, acoustics. 1136
Elasticity, plasticity 1143
Mathematical physics 1148
Optics, electromagnetic theory 1148

AUTHOR INDEX

Abdurahiman, P. V. 1140
Agarwal, Ratan Prakash 1084
Agostinelli, C. 1131, 1147
Ahlfors, L. V. 1050
Alexiewicz, A. 1070, 1093
Alexiewicz, A.-Orlicz, W. 1093
Alexits, G. 1081, 1083
Ament, W. S. 1149
Andreev, A. F. 1086
Angelo, E. J., Jr. 1130
Anis, A. A.-Lloyd, E. H. 1102
Aptê, A. 1138
Arbault, J. 1080
Arens, R. 1108
Aubert, K. E. 1056
Ayrault-Curie, G. 1114
Asbelev, N. V. 1129
Babukov, A. G. 1091
Bade, W. G. 1095
Baer, R. 1059
Bagchi, Hari das-Chaki, Manindra chandra 1116
Bagchi, Hari das-Mukherji, Bhola nath 1084
Bagchi, Hari das-Mukherji, Biswarup 1116
Bagchi, S. N. See Hosemann, R.
Bainda, E. 1089
Bailey, N. T. J. 1101
Bajraktarević, M. 1068
Bartels, R. C. F.-Laporte, O. 1141
Bartsch, H. 1119
Basch, A. 1051
Basu, D. 1103
Bateman, E. H. 1126
Beckenbach, E. F.-Jackson, L. K. 1084
Behlendorff, E. 1146
Belgrano Brémard, J. C. 1130
Bendukidze, A. D. 1079
Berezina, I. Ya. 1120
Bhate, D. H. 1102
Bieberbach, L. 1074
Birkhoff, G.-Kotik, J. 1137
Bishop, R. E. D. 1143
Bjerhammar, A. 1127
Blanuša, D. 1122
Bliss, C. I. 1102
Blum, E. K. 1096
Boas, R. P., Jr. 1074
Bochner, S. 1101
Bodewig, E. 1128
de Boer, J. H. 1117
Bolotin, V. V. 1146
Bompiani, E. 1122
Bondi, H.-Lyttleton, R. A. 1131
Bonnor, W. B. 1133
Borel, A. 1077
Borkmann, K. 1084
Bottema, O. 1114
Bourbaki, N. 1106
Bouwkamp, C. J. 1148
Brafman, F. 1083
*Braithwaite, R. B. 1097
Brauer, A. 1054, 1064
Braumann, P. B. T. 1053
Braunbek, W. 1136
Brehovskih, L. M. 1142
Bremekamp, H. 1085
Brennan, J. G. 1116
Britten, K. H. V. 1138
Bronkhorst, P. 1063
Brownlee, K. A.-Hodges, J. L., Jr.-Rosenblatt, M. 1104
*Bruins, E. M. 1049
Brusotti, L. 1050
Buchdahl, H. A. 1133
Buckens, F. 1146
Budden, K. G. 1083
Buharinov, G. N. 1145
Burau, W. 1117
Bureau, F. 1091
Burger, A. P. 1128
Busemann, H. 1115
Bykov, Ya. V. 1092
Čahtauri, A. I. 1120
Calapso, R. 1049, 1121
Calderon, A. P. 1071
Cantelli, F. P. 1098
Carin, V. S. 1059
Carlitz, L. 1064
Carstoiu, I. 1137
Carton, M. 1120
Castoldi, L. 1136
Cecconi, J. 1072
Čekmarev, A. I. See Kac, A. M.
Chaki, Manindra chandra. See Bagchi, Hari das.
Chalk, J. H. H. 1065
Chandrasekhar, S.-Elbert, D. 1126
*Chandrasekharan, K.-Minakshisundaram, S. 1077
Cheng, M. T. 1081
Cherep, R. 1121
Cherubino, S. 1095
Chung, Kai Lai 1099
Cinquini, S. See Cinquini Cibrario, M.
Cinquini Cibrario, M.-Cinquini, S. 1089, 1094
Clagett, M. See Moody, E. A.
Clemmow, P. C. 1149
Clemmow, P. C.-Senior, T. B. A. 1083
Clowes, J. S.-Hirsch, K. A. 1060
Clunie, J. 1078
Cohen, A. C. Jr. 1103
Cohen, K. J. 1053
Cohn, H. 1066
Colmez, J. 1106
Conway, H. D.-Huang, M. K. 1144
Cooke, R. G. 1093
Cordes, H. O. 1096
Coutoumanos, A. 1120
Cowling, T. G. 1130
Cox, D. R.-Smith, W. L. 1105
Coxeter, H. S. M.-Todd, J. A. 1066
Craig, W. 1051
Dalla Volta, V. 1121
Danskin, J. M. 1067
Dantu 1143
Das, A. C. 1103
Davenport, H.-Erdös, P. 1063
Davies, E. T. 1122
Davis, C. 1055
Davis, M. 1052
Davis, R. L. 1053
Dean, W. R. 1139
Deaux, R. 1050
Debrunner, H. 1115
Deheuvels, R. 1109
Delavault, H. 1090
Delerue, P. 1082
Delone, B. N. 1058
Denjoy, A. 1069
De Sloovere, H. 1150
Des Raj 1103
Deverall, L. I.-Thorne, C. J. 1144
Díaz, J. B. 1084
Dickinson, A. 1107
Dienemann, W. 1140
Dieudonné, J. 1062, 1093
Dijksterhuis, E. J. 1050
Dobrescu, A. 1062
Douglas, J. 1058
Dresher, M. 1067
Drobot, S. 1050
Dufresne, P. 1098
Dumas, M. 1102
Dunn, D. W.-Lin, C. C. 1141
Dwass, M. 1102
Dynkin, E. B. 1098
Dvaršelišvili, A. G. 1080
Eckart, C. 1142
Edmonds, S. M. 1081
Efremović, V. A. 1106
Egorov, D. 1051
Eisenhart, L. P. 1123
Ekstein, M. G. See Parrish, W.
Elbert, D. See Chandrasekhar, S.

(Continued on cover 3)

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HISTORY

Lacombe, Daniel. L'axiomatisation des mathématiques au III^e siècle avant J. C. Thalès 6 (1949–1950), 37–58 (1951).

It is the aim of the article under review to give (guided by the methods of modern axiomatics) a detailed investigation of the axioms in Euclid's Elements and in the Works of Archimedes. In this review it will suffice to mention some of the results obtained by the author in connexion with Archimedes' Lemma, the fifth axiom in the book, On the Sphere and Cylinder. This lemma is used by Archimedes in determining the volume of certain solids and the area of certain surfaces. There is, however, a difficulty inherent in this axiom, because mention is made of the difference between two magnitudes A and B, and it is by no means clear what this difference means if, e.g., A is the surface of a sphere and B is the surface of a cone. J. Hjelmslev called attention to this difficulty [Danske Vid. Selsk. Mat.-Fys. Medd. 25, no. 15 (1950); these Rev. 11, 571] and made it clear that Archimedes' Lemma contains more than is usually assumed. Without knowing this paper Lacombe mentions the same difficulty and deals with it from a somewhat different point of view. He emphasizes that in a modern treatment of length, area, and volume of lines, surfaces, and solids, respectively, the notion of point set is inevitable; this concept, however, was completely unknown, or the use of it was prohibited, in antiquity, probably because it appeared, for instance from Zeno's paradox, that the use of infinite point sets leads to insuperable difficulties. Lacombe now shows that by considering lines, surfaces, and solids as point sets for which he defines an equivalence relation satisfying the conditions $A_1, \cdots, A_5$ (p. 54) and by splitting Archimedes' Lemma into two, namely B_1 and B_2, it is possible to overcome the difficulties inherent in Archimedes' Lemma.

On pp. 56–58 it is shown that Archimedes tacitly, but no doubt consciously, avoids making the mistake, occurring in Euclid's Elements Book XII, of assuming the existence of a fourth proportional to three given magnitudes. Archimedes uses instead theorem 2, On the Sphere and Cylinder. This observation throws new light on this rather strange theorem. Footnote (1) p. 57 is by and large equivalent to a remark by J. Hjelmslev on p. 7 in the paper quoted above.

O. Schmidt (Copenhagen).

***Bruins, E. M. Nouvelles découvertes sur les mathématiques babyloniennes.** Université de Paris, Paris, 1952. 29 pp.

***The medieval science of weights. (Scientia de ponderibus.) Treatises ascribed to Euclid, Archimedes, Thabit ibn Qurra, Jordanus de Nemore, and Blasius of Parma.** Edited by Ernest A. Moody and Marshall Clagett. The University of Wisconsin Press, Madison, Wis., 1952. x+438 pp. $5.00.

This book contains eight treatises of the thirteenth and fourteenth centuries on weights, namely, Liber Euclidis de Ponderoso et Levi et Comparatione Corporum ad Invicem, Liber Archimedis de Insidentibus in Humidum (Liber Archimenidis de Ponderibus), Liber de Canonio, Liber Karastonis, Elementa Jordani super Demonstrationem Ponderum, Liber Jordani de Ponderibus, Liber Jordani de Nemore de Ratione Ponderis, and Tractatus Blasii de Parma de Ponderibus. Included also are four appendices: A Fragment of the Roman Balance Attributed to Euclid, Thomas Bradwardine's Discussion of Proposition One of the Liber de Ponderibus, A Fourteenth-century Commentary on Proposition One of the Liber de Ponderibus, and A Variant Form of Proposition Eight of the Elementa Jordani.

The Latin text and English translation of each of these selections are given on facing pages in a manner similar to the Loeb Classical Library editions of Greek and Latin classics. Each text is accompanied by a rather extended introduction. A list of manuscripts and editions of the texts is furnished and a section of the book is devoted to variant readings of the texts. A general introduction discusses the treatises and their place in the history of mechanics. Another portion of the book contains notes on each of the texts and on their introductions. A bibliography and an index are included.

The texts indicate that substantially the principle of virtual work was used in certain theorems in mechanics. Among other interesting features may be mentioned a correct solution of the problem of the inclined plane, usually credited to Galileo, and the use, on one occasion, of fairly advanced algebraic procedures. For those interested in the history of science, the texts are valuable as links in the transmission of Greek and Arabic science to the West. The bibliographical notes should be very helpful in locating material of importance to this history.

A few minor misprints were noted. *E. B. Allen.*

***Calapso, Renato. Matematici di Sicilia.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 274–286. Casa Editrice Perrella, Roma, 1953.

Sergescu, P. Quelques aspects de la vie scientifique en Italie au temps de la Renaissance. Rev. Gén. Sci. Pures Appl. 60, 101–113 (1953).

***Natucci, Alpinolo. La teoria delle proporzioni nel Rinascimento italiano.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 646–662 Casa Editrice Perrella, Roma, 1953.

Natucci, A. Le origini della teoria delle funzioni ellittiche. (Nel I° centenario della morte di Carlo Gustavo Jacobi 18 febbraio 1851.) Atti Accad. Ligure 9 (1952), 40–54 (1953).

Freudenthal, Hans. Zur Geschichte der vollständigen Induktion. Arch. Internat. Hist. Sci. 6, no. 22, 17–37 (1953).

*Taton, René. **La géométrie projective en France de Desargues à Poncelet.** Université de Paris, Paris, 1951. 21 pp. 60 francs.

*Ahlfors, Lars V. **Development of the theory of conformal mapping and Riemann surfaces through a century.** Contributions to the theory of Riemann surfaces, pp. 3–13. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

*Timoshenko, Stephen P. **History of strength of materials. With a brief account of the history of theory of elasticity and theory of structures.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. x+452 pp. $10.00.

The author's numerous earlier books are distinguished by their presentations based on original sources and by their numerous and careful attributions. In this volume he publishes in extended form his historical lectures. The main previous works on the subject are Todhunter and Pearson's "History of the theory of elasticity and of the strength of materials" [2 vols., Cambridge, 1886, 1893] and St. Venant's notes added to Navier's lectures ["Résumé des leçons . . . ," 3rd ed., Dunod, Paris, 1864] and to Clebsch's treatise ["Théorie de l'élasticité des corps solides," Dunod, Paris, 1883]. The author has read these works with care, but, besides including a great deal of additional material both from earlier periods and from later researches down to 1950, he has studied all sources anew, giving in some cases reasons for dissenting from accepted views. It is evident that this book is the result of great love and understanding for mechanics combined with many years of study and criticism. About 700 authors, many of whom are still living, are mentioned, references (often, unfortunately, incomplete and sometimes inaccurate) being given in each case to their relevant works; a small amount of biographical material, sometimes supplemented by a portrait, is supplied for about 70.

Since the subject is in part outside mathematics and in part on its borders, this review will not attempt to summarize the book in detail. For mathematicians already adept in continuum mechanics it will be a very useful and enjoyable work. First, it explains in historical setting connections between mathematical theory, engineering calculation or hypothesis, and test or experiment. Second and more important, it explains the guesses and practical expedients which must be used in domains of solid mechanics where adequate mathematical theory has yet to be constructed, whence the mathematical reader will carry away a hearty respect for the difficulty of the problems and the ingenuity with which engineers have faced them. For mathematicians who enjoy not only the solution but also the formulation of mathematical problems, this book with its critical summaries of discussions pro and con as practice has arisen historically will suggest far more challenges than the often rather pontifical engineering textbook treatments. It is interesting as a side light to learn about the engineering work of such mathematicians as Lamé and Klein. The author writes several illuminating discussions of mathematical instruction for engineers in various famous institutions at various periods and of the relation between mathematical knowledge and progress in engineering. The reader is led to infer from numerous examples that, at least in the past, a sound training in advanced mathematics has often led to engineering discoveries even of the most practical type, and that when the training of practical engineers at a place and time has neglected proper foundation in mathematics, the whole practice of design and testing has stood still or degenerated. While the author in his last three chapters carries the work up to the present day, he leaves the reader free to form his own judgment of this period.

The author's discussion of work before 1823 is somewhat misleading because he often explains it in terms of the concept of stress, which had not yet originated. It is also rather incomplete. The selection of subjects and authors since 1920 seems somewhat capricious to this reviewer. The author's presentation tends to emphasize the details at the expense of the principles. For example, there is little or no study of the origin of the concepts of shear strain and shear stress. Despite a choppy style and occasional errors in grammar and spelling, the strongly positive character of this book makes it difficult to put down until the last page is reached.

C. Truesdell (Bloomington, Ind.).

*Picone, Mauro. **Sull'opera matematica dell'Istituto Nazionale per le Applicazioni del Calcolo nel decorso quarto di secolo della sua esistenza.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 27–44. Casa Editrice Perrella, Roma, 1953.

*Kagan, V. F. **Arhimed. Kratkiĭ očerk o žizni i tvorčestve.** [**Archimedes. Short outline of his life and work.**] 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 56 pp. (2 plates). .80 rubles.

Popken, J. In memoriam: Prof. Dr. J. A. Barrau. Nieuw Arch. Wiskunde (3) 1, 89–91 (1953). (Dutch)

Procissi, Angiolo. Sui "Ragionamenti d'algebra" di Raffaello Canacci. Atti Accad. Ligure 9 (1952), 55–76 (1953).

Labrador, Juan Francisco. Juan Gaston Darboux. Gaceta Mat. (1) 5, 3–5 (1 plate) (1953). (Spanish)

Deaux, R. Obituary: Maurice Deweck (1904–1953). Mathesis 62, 81–84 (1 plate) (1953).

Brusotti, Luigi. Gli "Elementa" di Carlo Edoardo Filippa, allievo di Girolamo Saccheri. Atti Accad. Ligure 9 (1952), 155–164 (1953).

Pleijel, Åke. Nils Erik Fremberg in memoriam. Nordisk Mat. Tidskr. 1, 7–9 (1953). (Swedish)

*Galilei, Galileo. **Opere.** A cura di F. Flora. Riccardo Ricciardi Editore, Milano-Napoli, 1953. xxix+1139 pp.

Tenca, Luigi. Guido Grandi e i fondatori del calcolo infinitesimale. Boll. Un. Mat. Ital. (3) 8, 201–204 (1953).

Drobot, S. The scientific work of M. T. Huber. Zastosowania Mat. 1, 55–65 (1953). (Polish)

Dijksterhuis, E. J. Christiaan Huygens. Centaurus 2, 265–282 (1953).

Cf. these Rev. 14, 343.

al-Kāšī, Ğamšīd B. Mas'ūd. Der Lehrbrief über den Kreisumfang (ar-risāla al-muḥīṭīya). Übersetzt und erläutert von P. Luckey. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. 1950, no. 6, viii+95 pp. (1953).

This is a critical edition of a 15th century computation resulting in an approximation of 2π to ten sexagesimal places, a landmark of Islamic computational technique. The method is close to that of Archimedes, but more convenient for purposes of calculation.

Expressed in modern notation, al-Kāšī uses the relation $s_n=(2-(2+\cdots(2+1,0)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$ (n square roots) for the side of the regular polygon of $2^n\cdot 3$ sides inscribed in a circle of radius 1, $0=60$. He shows that putting $n=28$ insures the precision he demands, although the editor and translator demonstrates that considerable computation could have been dispensed with without affecting the accuracy of the result.

In the preparation of this edition only one manuscript [Istanbul Military Museum, No. 756] was available. The Mashhad Shrine Library copy, for instance, is reported [M. Ṭabāṭabā'ī, Āmūzeš va Parvareš (Persian), 10, 1–16] written in al-Kāšī's hand. However, the mathematical content of the work is clear without collation of other manuscript versions. The Arabic original with translation is complete except for the omission of parts of the computation which contribute nothing to the understanding of the work.

The paper of Ṭabāṭabā'ī cited above gives considerable previously unpublished biographical material on al-Kāšī.

E. S. Kennedy (Providence, R. I.).

***Štraĭh, S. Ya. Alekseĭ Nikolaevič Krylov, ego žizn' i deyatel'nost'. [Alekseĭ Nikolaevič Krylov, his life and works.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 88 pp. 1.40 rubles.

***Natucci, Alpinolo. Guglielmo Libri come storico della matematica.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 663–673. Casa Editrice Perrella, Roma, 1953.

Keldyš, L. V., and Novikov, P. S. The work of N. N. Luzin in the domain of the descriptive theory of sets. Uspehi Matem. Nauk (N.S.) 8, no. 2(54), 93–104 (1953). (Russian)

A list of Luzin's works on the descriptive theory of sets is included.

Egorov, D. Statement on N. N. Luzin's dissertation, "The integral and trigonometric series," presented to obtain the degree of Master of Pure Mathematics. Uspehi Matem. Nauk (N.S.) 8, no. 2(54), 105–110 (1953). (Russian)

This is the evaluation of Luzin's thesis [Moscow Univ., 1916] by one of his "official opponents". After discussing the work, Egorov recommends that Luzin be granted the degree of Doctor of Pure Mathematics, as was actually done by the faculty.

Basch, Alfred. Richard von Mises zum 70. Geburtstag. Österreich. Ing.-Arch. 7, 73–76 (1953).

Steck, Max. Proklus Diadochus und seine Gestaltlehre der Mathematik. Nova Acta Leopoldina (N.F.) 13, 131–149 (1943).

Der Verfasser behauptet dass die "Gestaltsynthese" von Proklus "in den deutschen Geistraum" in die Mathematiker Cusanus, Kepler und Leibniz, und in den durch Kant bedingten Idealismus eines Fichte weiter ausstrahlt. Sie findet "eine einmalige und bisher letzte Prägung grossen Stils bei Goethe." Diese "Gestaltsynthese" von Proklus wird etwa in folgender Weise definiert: "ein Gleichmass der Wiederholung gleichartiger Gestaltelemente, die der Geist als Ordnungselemente bereitstellt und so anordnet dass das an Zusammenhang, was ihm selbst gleicht, seinen vollkommensten und adäquatesten Ausdruck findet." Diese Art von Philosophie bildet eine Einleitung zu der Schönberger'schen Proklus Übersetzung [Deutsche Akademie der Naturforscher, Halle (Saale), 1945; diese Rev. 10, 173, 855]. Die Literaturhinweise sind wertvoll. *D. J. Struik.*

Puig Adam, Pedro. Torres Quevedo.—Mechanical calculation and automatism. Revista Acad. Ci. Madrid 47, 11–28 (1953). (Spanish)

Gonçalves, J. Vicente. Les formules de Ruffini. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 1, 408–409 (1951).

***Terracini, Alessandro. I quaderni di Corrado Segre.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 252–262. Casa Editrice Perrella, Roma, 1953.

Pignedoli, Antonio. Alcune considerazioni su Leonardo da Vinci artista, pensatore, scienziato. Accad. Sci. Modena. Atti Mem. (5) 10, 120–133 (1952).

FOUNDATIONS

***Langer, Susanne K. An introduction to symbolic logic.** 2d ed. Dover Publications, Inc., New York, N. Y., 1953. 367 pp. Cloth $3.50; paper $1.60.

Reprint of the 1st edition [Houghton Mifflin, Boston-New York; Allen and Unwin, London, 1937)].

***Kalmár, László. Reduction of the decision problem to the satisfiability question of logical formulae on a finite set.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 163–190. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and English summaries)

An English version of this paper is published in Acta Math. Acad. Sci. Hungar. 2, 125–142 (1951); these Rev. 14, 713.

***Hermes, H. Sur le concept d'axiomatisabilité.** Les méthodes formelles en axiomatique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 26, Paris, 1950, pp. 23–25; discussion, p. 26. Centre National de la Recherche Scientifique, Paris, 1953.

Essentially the same as Math. Nachr. 4, 343–347 (1951); these Rev. 12, 578.

Craig, William. On axiomatizability within a system. J. Symbolic Logic 18, 30–32 (1953).

Mit Hilfe eines elegant bewiesenen Satzes über rekursivaufzählbare Mengen wird für formale Systeme S und T, bei denen T alle Axiome und Ableitungsregeln von S enthält, gezeigt: die Menge C der Sätze von T, die Formeln von S sind, ist in S axiomatisierbar, d.h. es gibt eine primitiv-

rekursive Menge A von Formeln von S, so dass C die Menge der Formeln ist, die in S aus A ableitbar sind. Es werden wichtige Anwendungen dieses Satzes angeführt.

P. Lorenzen (Bonn).

***Feys, R. La formalisation comme suggestion rigoureuse. Les méthodes formelles en axiomatique.** Colloques Internationaux du Centre National de la Recherche Scientifique, no. 26, Paris, 1950, pp. 53–58. Centre National de la Recherche Scientifique, Paris, 1953.

McNaughton, Robert. Some formal relative consistency proofs. J. Symbolic Logic **18**, 136–144 (1953).

Let T_1 be the system Z of number theory described on p. 371 of Hilbert-Bernays. The system T_n, $n>1$, is formed as follows: Take as primitive atomic wffs first those of Z and consider the variables there to be of type 1, then add the formulas '$x_i \varepsilon y_{i+1}$', where i is a type subscript ($i \geqq 1$). T_n has further axiom schemata

A: '$(x_1)(y_1)(x_1 = y_1 \supset .F(x_1) \equiv F(y_1))$',
B: '$F(0).(x_1)(F(x_1) \supset F(x_1+1)).\supset (x_1)F(x_1)$',

where '$F(x_1)$' can be any wff of T_n. Identity is primitive for variables of type 1. For variables of higher type it is defined by coextensionality. T_n also has axiom schemata

C: '$x_i = y_i \supset (z_{i+1})(x_i \varepsilon z_{i+1} \equiv y_i \varepsilon z_{i+1})$' for $i<n$,
D: '$(\exists y_{i+1})(x_i)(x_i \varepsilon y_{i+1} \equiv F(x_i))$', for each $i<n$,

where '$F(x_i)$' is any wff of T_n in which 'y_{i+1}' is not free. The system I_n is like T_n, except that, in B, '$F(x_i)$' must have no variables of type greater than 1, and, in D, '$F(x_i)$' must, for each i, contain no variables of type greater than $i+1$. The consistency of I_n can be proved in T_3 and the consistency of I_ω in T_4, but T_2 and I_2 are of the same strength. If S and S' are two systems related as Quine's NF to the system described by H. Wang [same J. **15**, 25–32 (1950); these Rev. **11**, 636], then the consistency of S' relative to S can be proved in T_3. Moreover, the consistency of the von Neumann-Bernays set theory to the Zermelo-Fraenkel set theory can be proved in T_3. *I. Novak Gál.*

Davis, Martin. Arithmetical problems and recursively enumerable predicates. J. Symbolic Logic **18**, 33–41 (1953).

Ein Prädikat $(Ex_1, \cdots, x_n)p = q$ mit Polynomen p, q (die Koeffizienten seien natürliche Zahlen) heisse "diophantisch". Verf. beweist, dass jedes rekursivaufzählbare Prädikat $R(x_1, \cdots, x_n)$ als $(Ey)(k)(k \leqq y \rightarrow D(k, x_1, \cdots, x_n, y))$ darstellbar ist. Es bleibt offen, ob R stets diophantisch ist.

P. Lorenzen (Bonn).

Hasenjaeger, G. Eine Bemerkung zu Henkin's Beweis für die Vollständigkeit des Prädikatenkalküls der ersten Stufe. J. Symbolic Logic **18**, 42–48 (1953).

Durch eine Vereinfachung der Henkin'schen Konstruktion [derselbe J. **14**, 159–166 (1943); diese Rev. **11**, 487] von Modellen M für wf. Mengen Λ von Formeln des Prädikatenkalküls gelingen Verf. Abschätzungen der Kleene-Mostowski'schen Klasse von M. In der Darstellung von M treten nur 2 Quantoren mehr auf als in der von Λ.

P. Lorenzen (Bonn).

Izumi, Yosihisa. Über den Begriff der ω-Vollständigkeit. Tôhoku Math. J. (2) **4**, 314–315 (1952).

The author transforms a theorem of Tarski [Monatsh. Math. Phys. **40**, 97–112 (1933), p. 106. Satz 4] by the statement calculus. The proof of the second result contains an elementary mistake. *A. Heyting* (Amsterdam).

Rose, Alan. A formalization of an $\aleph_0$-valued propositional calculus. Proc. Cambridge Philos. Soc. **49**, 367–376 (1953).

Als "Wahrheitswerte" werden die rationalen Zahlen x mit $0 \leqq x \leqq 1$ genommen und "Wahrheitsfunktionen" N, H, und A definiert durch $\omega Np = 1 - \omega p$ (p, q als "Aussagenvariable", $\omega\mathfrak{A}$ bedeutet den Wahrheitswert des Ausdruckes $\mathfrak{A}$), $\omega Hp = \frac{1}{2}\omega p$, $\omega Apq = \min(1, \omega p + \omega q)$. Die Łukasiewicz'sche "Implikation" C ist dann definierbar durch $Cpq = ANpq$. Verf. löst die schwierige Aufgabe, einen Kalkül aufzustellen, indem genau die Identitäten, d.h. die Ausdrücke $\mathfrak{A}(p, q, \cdots)$ mit $\omega\mathfrak{A}(p, q, \cdots) \equiv 1$, ableitbar sind. Der Kalkül enthält 21 Axiome und neben p, $Cpq \rightarrow q$ noch eine weitere (komplizierte) Regel. *P. Lorenzen* (Bonn).

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. II. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 459–467 (1952).

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. III. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 1–11 (1953).

In this continuation of an earlier paper [same Proc. **55**, 213–223 (1952); these Rev. **14**, 527] the author discusses partially ordered systems with a closure operation in relation to various systems of modal logic. The modal systems in this paper are described by axioms which make apparent an analogy between the possibility operator and a topological closure operation. The methods depend upon unpublished work of Lindenbaum and are similar to those of McKinsey [J. Symbolic Logic **6**, 117–134 (1941); these Rev. **3**, 290]. The work affords a decision procedure for modal logics corresponding to Lewis's S_2 and S_4, for the author's system $K^{(\varphi)}$ intermediate to these, for a system corresponding to Lewis's S_3, as well as for systems closely related to these. As McKinsey observed in his paper, the decision procedure is of such complexity as to make its practical application questionable. *D. Nelson.*

Henkin, Leon. Some interconnections between modern algebra and mathematical logic. Trans. Amer. Math. Soc. **74**, 410–427 (1953).

Wie zuerst von Tarski entdeckt wurde, gestattet das Gödel'sche Vollständigkeitstheorem in der folgenden Form: "Ist für eine Menge Γ von Aussagen einer Sprache (die im elementaren Prädikatenkalkül mit beliebig vielen individuellen Konstanten formuliert sei) jede endliche Untermenge erfüllbar, dann ist Γ erfüllbar", algebraische Anwendungen. Z.B., gibt es beliebige grosse endliche Modelle von Γ, dann gibt es ein unendliches Modell; gibt es ein Modell mit unendlicher Kardinalzahl, dann gibt es Modelle für jede grössere Kardinalzahl.

Verf. verallgemeinert das Vollständigkeitstheorem auf Sprachen, die in Prädikatenkalkülen höherer Ordnung formuliert sind. An Stelle der Erfüllbarkeit tritt dann die allgemeine Erfüllbarkeit durch ein Modell, das aus einer Menge von Individuen und Familien von Klassen, von Klassen von Klassen, usw. besteht (also kein Standardmodell zu sein braucht). Auch für dieses allgemeine Vollständigkeitstheorem wird eine interessante Anwendung gegeben. *P. Lorenzen* (Bonn).

Henkin, Leon. Some notes on nominalism. J. Symbolic Logic **18**, 19–29 (1953).

Die "nominalistische" Auffassung von Quine [derselbe J. **12**, 74–84 (1947); diese Rev. **9**, 261] und Goodman und Quine [ibid. **12**, 105–122 (1947); diese Rev. **9**, 262] wird vom Verf. durch eine Erläuterung seiner Methode [ibid. **15**, 81–91 (1950); diese Rev. **12**, 70] unterstützt, wie die Prädikatenkalküle zweiter und höherer Ordnung interpretiert werden können, indem für die Prädikatenvariablen ein geeigneter Variabilitätsbereich sprachlicher Ausdrücke konstruiert wird. Die hierbei benutzte Unendlichkeit des Bereichs der Ausdrücke wird vom Verf. als eine "pretension" aufgefasst, die durch die evtl. Endlichkeit der Welt nicht ungültig würde. *P. Lorenzen* (Bonn).

Gilmore, P. C. The effect of Griss' criticism of the intuitionistic logic on deductive theories formalized within the intuitionistic logic. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 162–174, 175–186 (1953).

Es sei H eine deduktive Theorie mit den primitiven Aussagenformen $p, q, \cdots$, die in der intuitionistischen Logik formalisiert ist. Die Griss'sche Kritik verbietet das Auftreten von $\neg$ und gestattet ferner nur die Verwendung von Ausdrücken P, für die $(Ex_1 \cdots x_n)P$ beweisbar ist. Verf. gibt einen Prädikatenkalkül an, der diesen Forderungen genügt. An Stelle von H tritt dann eine deduktive Theorie GG', in der als primitive Aussageformen p nur diejenigen von H auftreten, für die $(Ex_1 \cdots x_n)p$ in H beweisbar ist. Innerhalb von GG' wird eine Teiltheorie definiert, die äquivalent ist zu derjenigen Teiltheorie von H, die entsteht, wenn als primitive Aussageformen nur diejenigen zugelassen werden, für die $(Ex_1 \cdots x_n)p \vee (Ax_1 \cdots x_n)\neg p$ beweisbar ist. Zur Definition einer Abbildung aus H in GG' wird benutzt, dass GG' eine primitive Aussageform $q(u, v)$ (z.B. $u \neq v$) enthält mit $(Eu, v)q(u, v)$ und $(Au)\neg q(u, u)$. Die Bedeutung dieser Äquivalenz für die Griss'sche Kritik wird ausführlich diskutiert. *P. Lorenzen* (Bonn).

***Gilmore, Paul Carl. The effect of Griss' criticism of the intuitionistic logic on deductive theories formalized within the intuitionistic logic.** Thesis, University of Amsterdam, 1953. ii+24+i pp. (Dutch summary)

Identical with the paper reviewed above.

Cohen, Kalman Joseph. A remark on Łukasiewicz's "On the intuitionistic theory of deduction". Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 111–112 (1953).

Łukasiewicz, Jan. Comment on K. J. Cohen's remark. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 113 (1953).

Łukasiewicz's paper [same Proc. **55**, 202–212 (1952); these Rev. **14**, 4] contains the result that every thesis of the intuitionistic calculus which contains F, T, or O only, remains true if we replace these functors by C, K, A respectively. Cohen remarks that this is not always the case if the substitution is carried out only for some occurrences of the functor. Łukasiewicz answers that a simpler counterexample was already contained in his paper.

A. Heyting (Amsterdam).

ALGEBRA

Mann, H. B., and Ryser, H. J. Systems of distinct representatives. Amer. Math. Monthly **60**, 397–401 (1953).

Soit $S_i \subseteq M$ $(i = 1, 2, \cdots, n)$ et $D = \{a_1, a_2, \cdots, a_n\}$ un système de représentants distincts (S.R.D.) des sous-ensembles S_i, c'est-à-dire tels que $a_i \in S_i$; $a_i \neq a_j$ si $i \neq j$; $(i, j = 1, 2, \cdots, n)$. Exemple:

$$S_1 = \{1, 2, 3\}; \quad S_2 = \{1\}; \quad S_3 = \{2, 3\}; \quad D = \{2, 1, 3\}.$$

Il est montré que: (1) Si k quelconques des S $(k = 1, 2, \cdots, n)$ contiennent ensemble au moins k éléments distincts de M, alors il existe un S.R.D. pour ces sous-ensembles. Et si r ne surpasse pas le nombre minimum d'éléments dans chaque S, alors il y a au moins $r!$ S.R.D. pour ces sous-ensembles si $n \geq r$, et $r!/(r-n)!$ si $n < r$. Ceci est un perfectionnement du théorème de P. Hall [J. London Math. Soc. **10**, 26–30 (1935)]. (2) Si les ensembles S_i satisfont aux conditions de (1) et si chacun des éléments $e_1, e_2, \cdots, e_k$ apparaît au moins t fois dans les ensembles S_i, en même temps que chaque S_i contient au plus t des éléments $e_1, e_2, \cdots, e_k$, alors il existe un S.R.D. contenant tous les e. (3) Si les S_i remplissent les conditions du théorème (1), si chacun des éléments $e_1, \cdots, e_k$ figure au moins t fois dans l'ensemble des S_i et si chaque S_i contient au plus t éléments, alors il existe, pour tout élément $\alpha \in M$, un S.R.D. contenant α et tous les e. Application aux matrices dont les éléments sont 0 et 1, aux balanced incomplete block designs symétriques [H. B. Mann, Analysis and design of experiments, Dover, New York, 1949; ces Rev. **11**, 262], aux groupes finis, aux rectangles latins, à la recherche d'une limite de leur nombre et à leur immersion dans un carré latin [H. J. Ryser, Proc. Amer. Math. Soc. **2**, 550–552 (1951); ces Rev. **13**, 98]. *A. Sade.*

Davis, Robert L. The number of structures of finite relations. Proc. Amer. Math. Soc. **4**, 486–495 (1953).

Für eine endliche Menge N stellen zwei m-stellige Relationen A, B von N dieselbe "Struktur" dar, wenn es eine Permutation π von N gibt mit

$$A(x_1, \cdots, x_m) \leftrightarrow B(\pi x_1, \cdots, \pi x_n)$$

für alle $x_1, \cdots, x_n$ aus N. Mit Hilfe der Zerlegung der Permutationen in Zykeln bestimmt der Verf. die Anzahl aller Strukturen. Für 2-stellige Relationen werden auch die Anzahlen der reflexiven symmetrischen u.ä. Strukturen berechnet. *P. Lorenzen* (Bonn).

Braumann, Peter Bruno Theodor. Partitions in various branches of mathematics. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1**, 205–296 (1951). (Portuguese. German summary)

The author considers properties of partitions $I\{i_0, \cdots, i_{l-1}\}$ of an integer I into integral, non-negative components i_k so that $I = i_0 + i_1 + \cdots + i_{l-1}$ and among these particularly the partitions with m elements where $l = m$ and $0 \leq i_k \leq m-1$ for all k. When $I = mq + r$ the numbers q and r are called the weight and remainder, respectively. Two partitions are congruent if they have the same remainder, complementary if $i_k + i_k' = m-1$ for every k. They are concyclic if $i_k' \equiv i_k + l \pmod{m}$. The smallest $l \neq 0$ for which a partition is concylic to itself is called its degree d. The partition is ordinary [primitive] if $m = d$, otherwise extraordinary. The author deduces various properties of the numbers $\phi_0(m)$ and $\phi_*(m)$ of primitive and imprimitive partitions of degree m and also the corresponding numbers for partitions of remainder 0.

To every partition of m elements the author associates a sum of roots of unity

$$(1)\quad \sigma(i_0, \cdots, i_{m-1}) = \sum \exp \frac{2\pi i}{m}(i_0 s_0 + \cdots + i_{m-1} s_{m-1})$$

where the sum is extended to all sets of integers $0 \leqq s_k \leqq m-1$ with $s_k \neq s_l$. Various properties of these numbers are derived and applications made to combinatorial problems and resultants for polynomials. [The numbers (1) are also the coefficients in the cyclic determinant of order m, for which the reviewer has given an explicit expression [Duke Math. J. 18, 343–354 (1951); these Rev. 13, 98].] *O. Ore.*

Tenca, Luigi. Osservazioni sui determinanti circolanti. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 15(84), 272–280 (1951).

Let n distinct quantities $a_1, \cdots, a_n$ be given. One may then form a cyclic determinant $\Delta(a_1 \cdots a_n)$ having these a_i as its first line. By permuting the a_i one obtains $n!$ different cyclic determinants. These determinants are divided into classes such that each class contains the determinant defined by a permutation of the a_i and all determinants resulting from it by a cyclic permutation and the inverse of the given permutation. All determinants in a class have the same value except possibly for sign. The author determines relations between the determinants of the various classes. *O. Ore.*

Brauer, Alfred. On a new class of Hadamard determinants. Math. Z. 58, 219–225 (1953).

An Hadamard determinant of order N is a determinant with elements ± 1 taking the maximal value $N^{N/2}$ possible according to Hadamard's determinant theorem. A variety of classes of such determinants has been found. The author gives a new class of order $p(p+2)+1$ and certain multiples of this order, provided p and $p+2$ form a twin pair of primes. The method is based upon certain theorems on quadratic residues. *O. Ore* (New Haven, Conn.).

Waadeland, Haakon. Über eine Determinante. Norske Vid. Selsk. Forh., Trondheim 24 (1951), 108–109 (1952).

Consider the determinant

$$D_n(x) = \det|a_{ij}(x)| \quad (i, j = 1, 2, \cdots, n-1), \quad n \geqq 2,$$

where $a_{ij}(x) = \{(i-j+1)n + (j-1)\}\binom{x}{i-j+2}$, the symbol $\binom{x}{k}$ being defined as $x(x-1)\cdots(x-k+1)/k!$ when k is a positive integer and as zero in all other cases. Jacobsthal has proved [same Forh. 23, 127–129 (1951); these Rev. 13, 98] that $D_n(x) = x^{n-1}(x-1)(2x-1)\cdots((n-1)x-1)$. The author gives a new proof for the value of $D_n(r)$, r a positive integer, by applying explicit known formulas for the inversion of a power series to $w = (1+z)^r - 1$. The value of $D_n(x)$ follows from that of $D_n(r)$. *G. B. Price.*

Gyires, Bela. Ein Determinantensatz. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 2, ~~no. 3-4 229~~ 463-474 (1951). (Romanian and German. Russian summary.)

A complicated generalization of Kronecker's determinant theorem is proved in a straightforward manner. Out of the matrices

$$N_{\nu_i\nu_j} = \begin{bmatrix} a_{kj}^{(1)} & b_{11}^{(j)} & \cdots & a_{kj}^{(1)} & b_{1\nu_j}^{(j)} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{kj}^{(\nu_i)} & b_{\nu_i 1}^{(j)} & \cdots & a_{kj}^{(\nu_i)} & b_{\nu_i\nu_j}^{(j)} \end{bmatrix}$$

$(k, j = 1, 2, \cdots, \mu)$ build the determinant of order

$$n = \nu_1 + \nu_2 + \cdots + \nu_\mu$$

$$D = \begin{vmatrix} N_{\nu_1\nu_1} \cdots N_{\nu_1\nu_\mu} \\ \cdot \quad \cdots \quad \cdot \\ N_{\nu_\mu\nu_1} \cdots N_{\nu_\mu\nu_\mu} \end{vmatrix}.$$

Let

$$A_{\mu_i}^{(i)} = \begin{bmatrix} a_{11}^{(i)} \cdots a_{\mu_i 1}^{(i)} \\ \cdot \quad \cdots \quad \cdot \\ a_{1\mu_i}^{(i)} \cdots a_{\mu_i\mu_i}^{(i)} \end{bmatrix}, \quad B_{\nu_j}^{(j)} = \begin{bmatrix} b_{11}^{(j)} \cdots b_{1}^{(j)}{}_{j} \\ \cdot \quad \cdots \quad \cdot \\ b_{\nu_j 1}^{(j)} \cdots b_{\nu_j\nu_j}^{(j)} \end{bmatrix}.$$

Suppose the numbers ν_j form a decreasing sequence $\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_l (\geqq 1)$, and μ_i denotes that index for which $\nu_{\mu_i} \geqq i > \nu_{\mu_i+1}$ $(i = 1, 2, \cdots, \nu_1 (= \nu))$. Theorem:

$$D = |A_{\mu_1}^{(1)}| \cdots |A_{\mu_\nu}^{(\nu)}| \cdot |B_{\nu_1}^{(1)}| \cdots |B_{\nu_\mu}^{(\mu)}|.$$

The paper is replete with misprints which have been corrected on the reviewer's copy. *G. E. Forsythe.*

Wong, Y. K. Some inequalities of determinants of Minkowski type. Duke Math. J. 19, 231–241 (1952).

The author establishes inequalities for the determinant D_n of a matrix of the form $I-a$, where I is the n by n identity matrix, and where a is an n by n matrix with elements $a(i, j)$. Let $\|a\|$ denote the Hilbert modulus of the matrix a; then $\|a\| = \sup\{\|ax\| : x \text{ a unit vector}\}$, where $\|x\|$ is the length of x. Consider the following properties: (P1) $0 \leqq a(i, j)$ for $i, j = Z, \cdots, n$; (P2) $\|a\| < 1$; (P3) $s^{(n)}(j) = \sum_{i=1}^{n} a(i, j) \leqq 1$ and $a(j, j) < 1$ for $j = 1, \cdots, n$. Let D_k, $k = 2, \cdots, n$, be the determinant of the principal minor $I - a_k$ which consists of the first k rows and columns of $I-a$. Lemma 1. If a satisfies (P1) and (P2), then $0 < D_k \leqq D_{k-1} \leqq 1$ for $k = 2, \cdots, n$. Lemma 2. If a satisfies (P1) and (P3), then the adjoint M_n of $L_n = I - a$ has only non-negative elements, and $D_{n-1} \geqq D_n \geqq 0$. Corollary 1. If a satisfies (P1) and (P3), and if $\sum_{j=1}^{k} a(j, k) < 1$ for $k = 1, \cdots, n$, then $D_n > 0$. Among other results and refinements, proofs are given, for the special determinants considered in this paper, for inequalities established by Price [Proc. Amer. Math. Soc. 2, 497–502 (1951); these Rev. 12, 793]. Finally, if a satisfies (P1), (P2), and (P3), the author shows that

$$[1 - a(n, n) - \sum_{j=1}^{n-1} s^{(n)}(j) a(j, n)] D_{n-1} \leqq D_n$$

$$\leqq [1 - a(n, n) - \sum_{i=1}^{n-1} a(n, i) a(i, n)] D_{n-1}.$$

For the special determinants to which they apply, these inequalities are better than similar ones given by Price. *G. B. Price* (Lawrence, Kan.).

Wong, Y. K. An inequality for Minkowski matrices. Proc. Amer. Math. Soc. 4, 137–141 (1953).

Let a be an n by n matrix with real or complex elements $a(i, j)$, and let D_n and $I - a_{n-1}$ have the same meaning as in the preceding review. Consider the following property: (P4) $s^{(n)}(j) = \sum_{i=1}^{n} |a(i, j)| \leqq 1$ and $|a(j, j)| < 1$ for $j = 1, \cdots, n$. Let M_{n-1} be the adjoint of the matrix $I - a_{n-1}$. Lemma. If $s^{(n)}(j) \leqq 1$ for $j = 1, \cdots, n-1$, then

$$\sum_{i=1}^{n-1} |a(n, i)| \cdot |M_{n-1}(i, k)| \leqq s^{(n)}(k) |D_{n-1}|, \quad k = 1, \cdots, n-1.$$

The proof of this inequality is based on weaker hypotheses than a previous proof given by the author [see the paper reviewed above]. The lemma is applied to prove that, if a satisfies (P4), then

$$[|1 - a(n, n)| - N^{(n)}] |D_{n-1}| \leqq |D_n|$$

$$\leqq [|1 - a(n, n)| + N^{(n)}] |D_{n-1}|,$$

where $N^{(n)} = \sum_{j=1}^{n-1} s^{(n)}(j) |a(j, n)|$. These inequalities are better than similar ones given by Price [see the reference in the preceding review] and Ostrowski [same Proc. 3, 26–30

(1952); these Rev. **14**, 611]. Finally, the author gives refinements of the result in the lemma and of the bounds for $|D_n|$. The proofs employ only elementary properties of determinants. *G. B. Price* (Lawrence, Kan.).

Schneider, Hans. An inequality for latent roots applied to determinants with dominant principal diagonal. J. London Math. Soc. **28**, 8–20 (1953).

Let λ_i be the latent roots of a matrix $A=(a_{ij})$. Frobenius showed that (*) $\max|\lambda_i| \leqq \max_i \sum_j |a_{ij}|$. Let X be the diagonal matrix diag $[x_1, \cdots, x_n]$, $x_i>0$. The inequality (*) is applied to $X^{-1}CX$, where $C=(|a_{ij}|)$, and is then generalized to an inequality for $\max_{i_r} \prod_{r=1}^{k}|\lambda_{i_r}|$. Conditions for equality are discussed. Bounds for determinants with dominant principal diagonals are obtained, improving bounds previously obtained by H. v. Koch, A. Ostrowski, and G. B. Price. Conditions are found under which a matrix is similar to a matrix with dominant principal diagonal when transformed by a diagonal matrix. *G. E. Forsythe.*

Riguet, Jacques. Matrices de Stirling. C. R. Acad. Sci. Paris **236**, 1839–1841 (1953).

If u is a sequence with values in an integral domain, $\underset{n}{u}$ is the sequence with the same first n elements, and all other elements zero. If v is a sequence with only a finite number of non-zero elements, then h_v is the sequence whose ith element is the ith-degree Wronski $\aleph$-function of $v(0), v(1), \cdots$. The author states a theorem to the effect that the Stirling matrix for the sequence u, i.e., the matrix whose (i, j)th element is $h_{\underset{j}{u}}(i-j)$, has a unique inverse whose (i, j)th element is $h_{\underset{i}{u}}^{-1}(i-j)$. *H. A. Thurston* (Bristol).

Kotelyanskiĭ, D. M. On some properties of matrices with positive elements. Mat. Sbornik N.S. **31(73)**, 497–506 (1952). (Russian)

Let $A=(a_{ij})$ be a matrix all of whose elements are positive, let its characteristic numbers be ρ_i; set $B=(b_{ij})=\lambda I-A$. Theorem I. A necessary and sufficient condition that λ satisfy $\lambda \geqq |\rho_i|$ is that the nested set of principal r^2 minors $\det(b_{jk})$, $1 \leqq j, k \leqq r$, $r=1, 2, \cdots, n$, of B should not be negative. (The necessity is due to Perron [Math. Ann. **64**, 1–76 (1907)].) Theorem II. If $b_{jj} \geqq 0$, and every principal $(n-2)^2$, $(n-4)^2$, etc. minor of B is $\geqq 0$, then $\lambda \geqq |\rho_i|$. Theorem III. If all cofactors of the first row in B are $\geqq 0$ and are not $\equiv 0$, then $\lambda > |\rho_i|$ ($\lambda < \max|\rho_i|$) if $\det B>0$ ($\det B<0$). Theorem IV. Suppose the cofactors of every element in B are positive. Let $A'=(a'_{ij})$ with $0<a'_{ij} \leqq a_{ij}$ $(i, j=1, \cdots, n)$. Then $\det(\lambda I-A')$, $1 \leqq i, j \leqq k$, is not less than the corresponding leading k^2 minor $\det(\lambda I-A)$, $1 \leqq i, j \leqq k$, of B. If every principal minor of order $1^2, \cdots, (n-2)^2$ of B is positive, then $\det B < \prod_i^n(\lambda-a_{ii})$.

J. L. Brenner (Pullman, Wash.).

Quade, W. Beweis eines Satzes über positive Matrizen. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 50–51 (1953).

Theorem [proved three times in Peremans, Duparc and Lekkerkerker, Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 24–27 (1952); these Rev. **14**, 125]: Let the symmetric matrix $C=A+iB$ (A, B real) be such that $u'Au \geqq 0$ for all real vectors u. Then, if $\det C=0$, there exists a real vector $s \neq 0$ such that $Cs=0$. The author gives a proof phrased in matrix notation.

G. E. Forsythe (Los Angeles, Calif.).

Stojakovic, Mirko. Sur les matrices quasi-inverses et les matrices quasi-unités. C. R. Acad. Sci. Paris **236**, 877–879 (1953).

The author announces some new properties of his recently defined inverse of a rectangular matrix A [Bull. Soc. Roy. Sci. Liège **21**, 303–305 (1952) and Bull. Soc. Math. Phys. Serbie **4**, nos. 1–2, 9–23 (1952); these Rev. **14**, 443], and compares his definition further with Bjerhammar's (non-unique) A^{-1} [Bull. Géodésique **1951**, 188–220; these Rev. **13**, 312]. If A has l rows and k columns and $l \leqq k$, then AA^{-1} is the unit matrix of order l, whereas the characteristic polynomial of $A^{-1}A$ is stated to be $\lambda^l(\lambda-1)^{k-l}$. Reviewer's query: should not this read $\lambda^{k-l}(\lambda-1)^l$, in order to reduce to $(\lambda-1)^k$ for $l=k$? *G. E. Forsythe* (Los Angeles, Calif.).

Rutishauser, Heinz. Beiträge zur Kenntnis des Biorthogonalisierungs-Algorithmus von Lanczos. Z. Angew. Math. Physik **4**, 35–56 (1953).

This is a detailed theoretical study of the practically important procedure [C. Lanczos, J. Research Nat. Bur. Standards **45**, 255–282 (1950); these Rev. **13**, 163] for transforming a real matrix A of order N into "codiagonal" form, in which $a_{ij}=0$ for $|i-j|>1$. The procedure effectively biorthogonalizes two sequences $x, Ax, A^2x, \cdots$, and $y, A^*y, A^{*2}y, \cdots$, where A^* is the transpose of A. One thus obtains sequences $\{x_i\}$ and $\{y_i\}$. The following are proved: (I) If the minimal polynomial of A is of degree m, one can choose x, y so the algorithm terminates in exactly m steps with $(x_m, y_m) \neq 0$. (II) If all the eigenvalues of A are real, the codiagonal elements $\beta_k=(Ax_{k+1}, y_k)/(x_k, y_k)$ can be made arbitrarily small by suitable choice of x, y. (III) If all $\beta_k>0$, precise bounds for the eigenvalues can be given in terms of the β_k and the $\alpha_k=(Ax_k, y_k)/(x_k, y_k)$. For a certain one-parameter family of trial vectors $x(l)$, $y(l)$, the α_k and the β_k satisfy a system of differential equations which are analogous to the Frenet formulas of geometry. There is much other material. *G. E. Forsythe.*

Davis, Chandler. The intersection of a linear subspace with the positive orthant. Michigan Math. J. **1** (1952), 163–168 (1953).

In a Euclidean space E^n, let P^n denote the set of vectors with all components non-negative. The author first shows that if (1) A is an $n \times k$ matrix with non-negative entries and (2) $AE^k \cap P^n \subset AP^k$, then (2′) $A'E^n \cap P^k \subset A'P^n$, where prime denotes transposition. It should be noted that the lemma used in the proof is a geometrical version of the "Grundsatz" of J. Farkas [J. Reine Angew. Math. **124**, 1–27 (1902), pp. 5–7] and that (1) may be dropped. The same lemma is used to show that, for matrices A satisfying (1) and (2), $A'P^n$ is affine isomorphic to the set of vectors in AE^k making a non-obtuse angle with all of AP^k (the "geometric polar" of AP^k). *H. W. Kuhn* (Bryn Mawr, Pa.).

***Faddeev, D. K., i Sominskiĭ, I. S. Sbornik zadač po vysšeĭ algebre.** [Collection of problems on higher algebra.] 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 308 pp. 7.20 rubles.

Part 1, Problems: I) Complex numbers; II) Computation of determinants; III) Systems of linear equations; IV) Matrices; V) Polynomials and rational functions of one variable; VI) Symmetric functions; VII) Linear algebra. Part 2, Hints. Part 3, Answers and solutions.

Table of Contents.

Abstract Algebra

***Zappa, Guido. La teoria dei reticoli e le sue applicazioni a vari rami della matematica.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 167–185. Casa Editrice Perrella, Roma, 1953.

Expository article, with especial emphasis on applications to group theory. *G. Birkhoff* (Cambridge, Mass.).

Elliott, J. G. Autometrization and the symmetric difference. Canadian J. Math. **5**, 324–331 (1953).

Soit E une algèbre de Boole dans laquelle la borne supérieure de deux éléments a, $b \in E$ est notée $a+b$ et dans laquelle leur différence symétrique est notée $a \oplus b$. Une opération binaire partout définie sur E notée $*$ sera dite métrique si elle satisfait aux trois axiomes: $M_1)$ $a*b=0 \leftrightarrow a=b$, $M_2)$ $a*b=b*a$, $M_3)$ $(a*b)+(b*c)>a*c$. Alors, si $*$ est une opération métrique et si $*$ est une opération de semigroupe ou une opération de quasigroupe ou une opération faiblement associative (c'est-à-dire, telle que $a*(a*b)=(a*a)*b$), $*$ est identique à $\oplus$. Les autres résultats contenus dans ce travail sont bien connus. *J. Riguet* (Paris).

Aubert, Karl Egil. Sur le radical de McCoy. C. R. Acad. Sci. Paris **237**, 10–12 (1953).

Let L be a Boolean algebra with a multiplication having the properties listed in an earlier paper [same C. R. **236**, 31–33 (1953); these Rev. **14**, 529] except for the commutative law. Prime ideals and the prime radical of McCoy [Amer. J. Math. **71**, 823–833 (1949); these Rev. **11**, 311] are defined in L and are shown to have the expected properties. *R. E. Johnson* (Northampton, Mass.).

Zemmer, Joseph L. Ordered algebras which contain divisors of zero. Duke Math. J. **20**, 177–183 (1953).

Verlangt man von der Ordnung $>$ eines Ringes nicht $x>0$, $y>0 \rightarrow xy>0$, sondern nur $\rightarrow xy \geqq 0$, dann gibt es geordnete Ringe mit echten Nullteilern. In Verschärfung eines Resultats von A. Albert [Bull. Amer. Math. Soc. **46**, 521–522 (1940); diese Rev. **1**, 328] ist jede einfache geordnete Algebra endlicher Dimension über einem Körper selbst ein Körper. Verf. bestimmt darüber hinaus die Struktur der irreduziblen geordneten Algebren endlicher Dimension, soweit sie weder Körper noch nilpotent sind. Zum Schluss einige Bemerkungen über nilpotente geordnete Algebren. *P. Lorenzen* (Bonn).

Klingenberg, Wilhelm. Sopra il numero degli ordinamenti di un corpo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 395–396 (1953).

Theorem: Let K be a field which admits only archimedean orderings and let n be the number of such orderings. If Q^* is the group of nonzero sums of squares then the index of Q^* in the multiplicative group of K is 2^n. The proof consists in constructing n elements, each positive in exactly one ordering. Their 2^n products are then incongruent modulo Q^*. (Thus also if n is infinite, the index of Q^* is infinite.) *D. Zelinsky* (Evanston, Ill.).

Osima, Masaru. Note on inverses in rings. J. Gakugei Tokushima Univ. **3**, 21–23 (1953).

The author proves some results about rings possessing elements with right inverses but no left inverses. The most interesting of these results is one which is known, occurring in a paper by Jacobson [Proc. Amer. Math. Soc. **1**, 352–355 (1950); these Rev. **12**, 75]. *I. N. Herstein.*

***Szele, Tibor. Ein Zerfällungssatz für radikalfreie Ringe.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 429–434. Akadémiai Kiadó, Budapest, 1952. (Hungarian and German. Russian summary)

The author proves: let R be a ring without nilpotent ideals, and let L be a left ideal of R which possesses in it a left unit element; then R decomposes into a direct sum of two two-sided ideals, one of which is L. *I. N. Herstein* (Philadelphia, Pa.).

Nollet, Louis. Construction des anneaux dont tout sous-anneau est un idéal. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 632–636 (1952).

If a ring A possesses a left identity element and every subring of A is either a right or a left ideal, then A is isomorphic which a product $J \times G$ where: 1) J is the ring of rational integers E when A is of characteristic zero or E/q when A is of non-zero characteristic q; 2) G is an additive Abelian group of characteristic a divisor of q; 3) the elements of $J \times G$ are pairs (j, g) where $(j, g)+(j', g')=(j+j', g+g')$ and $(j, g)(j', g')=(jj', jg')$, jg' having meaning since the characteristic of G divides that of J. *F. Kiokemeister.*

Raffin, R. Immersion dans un domaine à division de l'anneau des classes résiduelles modulo n. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 957–964 (1952).

Let n be a rational integer. If a/b is a rational number, $\omega_n(a/b)$ is defined to be that rational number p/q arrived at as follows: a is replaced in a/b by its remainder a_r after division by nb. Then b is replaced in a_r/b by its remainder after division by na_r. The fraction a_r/b_r is treated in the same fashion. These operations are carried out until one of three cases arises. 1) One arrives at a rational number t/s for which $t \neq 0$, $s \neq 0$, and $t_r=t$, $s_r=s$. Then $p/q=t/s$. 2) There occurs at some stage the form $0/m$ with $m \neq 0$. Then $p/q=0$. 3) There occurs at some stage the formal fraction $m/0$. Then $p/q=1/n$.

Let x and y be rational numbers. Define $x+y=\omega_n(x+y)$ and $x \dot{\times} y=\omega_n(xy)$. The system D_n thus constructed has the properties: D_n is non-distributive; in D_n division by a non-zero element is always possible in an infinite number of ways; and D_n contains a subset isomorphic with the ring of rational integers modulo n. *F. Kiokemeister.*

Herstein, I. N. Finite multiplicative subgroups in division rings. Pacific J. Math. **3**, 121–126 (1953).

If G is a finite subgroup of the multiplicative group of nonzero elements of a commutative field, then it is known that G must be cyclic. The author gives several extensions of this statement to the case of skew fields. He shows that a finite abelian subgroup of the multiplicative group K^* of a skew field K is cyclic. Moreover, if the order of a finite subgroup G of K^* is odd, then G is necessarily metacyclic, and if the order of G is an odd prime power, then G is necessarily cyclic. If K is of characteristic $p \neq 0$, then every finite subgroup of K^* is cyclic. Finally, the author's general conjecture according to which any finite multiplicative subgroup of odd order of an arbitrary skew field K is cyclic is proved for the case in which K is the skew field of the quaternions over the real field. *T. Szele* (Debrecen).

Hua, Loo-Keng. A note on the total matrix ring over a non-commutative field. Ann. Soc. Polon. Math. **25** (1952), 188–198 (1953).

Concerned primarily with the Lie and Jordan ring structures of a matrix ring $\mathfrak{D}_n$ over a division ring $\mathfrak{D}$, the author

establishes by matrix calculations that for characteristic $\mathfrak{D} \neq 2$, (i) $\mathfrak{D}_n$ is a simple Jordan ring, (ii) every Lie ideal of $\mathfrak{D}_n$ not contained in the center contains $(\mathfrak{D}_n)'$, and (iii) the only proper Lie ideals of $(\mathfrak{D}_n)'$ are those contained in the center of $\mathfrak{D}_n$. The results are obtained together by studying additive subgroups of $\mathfrak{D}_n$ closed under $X \to A^{-1}XA + AXA^{-1}$ for every non-singular A in $\mathfrak{D}_n$. *W. G. Lister.*

Mal'cev, A. I. Multiplicative congruences of matrices. Doklady Akad. Nauk SSSR (N.S.) **90**, 333–335 (1953). (Russian)

Soient: F_n le système multiplicatif des matrices $n \times n$ à éléments dans un corps F, F_n^r le sous système multiplicatif de F_n constitué par les matrices $n \times n$ de rang $\leqq n$. Le groupe multiplicatif de F sera désigné par F^*. L'auteur établit ici le théorème suivant: Pour que la relation binaire R entre éléments de F_n^r soit une relation d'équivalence compatible avec la structure multiplicative il faut et il suffit qu'il existe une suite non décroissante $G_1 \subset G_2 \subset \cdots \subset G_{\mu-1}$ de sous groupes de F^* et une sous groupe distingué G_μ du groupe linéaire dans F_n tels que les matrices A et B de F_n^r soient dans la relation R si et seulement si l'une des trois conditions suivantes est remplie: (a) rang $A \leqq r-\mu$, rang $B \leqq r-\mu$; (b) rang $A = r-i+1$, et il existe $\lambda \in G_i$ $(\mu-1 \leqq i \leqq r)$ tel que $B = \lambda A$; (c) rang A = rang $B = r-\mu+1$, $LA = LB$, $B = AU$, $U_0 \in G_\mu$, où U est une matrice de rang $r-\mu+1$, où L est l'espace des vecteurs-lignes $1 \times n$, où U_0 est la matrice de la transformation induite par U dans le sous espace LA. On voit facilement que la condition est suffisante. La démonstration de la nécessité de la condition résulte de raisonnements assez analogues à ceux de l'auteur [Mat. Sbornik **31(73)**, 136–151 (1952); ces Rev. **14**, 349]. Ce résultat donne un dénombrement complet de toutes les équivalences compatibles sur F_n^r puisque, depuis Dickson, on connaît tous les sous groupes distingués du groupe linéaire. *J. Riguet* (Paris).

Goldhaber, J. K. Special types of linear mappings of algebras. Amer. J. Math. **75**, 91–97 (1953).

Let ψ be a linear mapping of an algebra $\mathfrak{A}$ over a field F into itself for which $\psi(\mathfrak{A}) \neq 0$. Let F be of characteristic not two. ψ will be an automorphism or an anti-automorphism of $\mathfrak{A}$ in each of the following cases. 1) If $\mathfrak{A}$ is a total matrix algebra and $\psi(A^2) = [\psi(A)]^2$ when $A^2 = A$ or $A^2 = 0$. 2) If $\mathfrak{A}$ is a central simple algebra, $\psi(I) = I$ where I is the identity of $\mathfrak{A}$, and $f[\psi(A)] = 0$ when f is the characteristic function of A. 3) If $\mathfrak{A}$ is semi-simple, the conditions of 2) are satisfied, and ψ is bi-unique.

These theorems generalise results of Ancochea [Ann. of Math. (2) **48**, 147–153 (1947); these Rev. **8**, 310], Hua [Proc. Nat. Acad. Sci. U. S. A. **35**, 386–389 (1949); these Rev. **10**, 675], and Kaplansky [Duke Math. J. **14**, 521–525 (1947); these Rev. **9**, 172]. *F. Kiokemeister.*

Herz, Jean-Claude. Pseudo-algèbres de Lie. I. C. R. Acad. Sci. Paris **236**, 1935–1937 (1953).

The notion of a pseudo Lie algebra over a division ring K is defined by replacing the relevant part of the axiom of bilinearity of the commutator in a Lie algebra with the axiom $[u, \gamma v] = \gamma[u, v] + \alpha v$, for all $\gamma \in K$, where α is an element of K which depends only on u and γ, and is accordingly denoted $u\gamma$. The map $\gamma \to u\gamma$ is then a derivation of K. A few simple examples of such structures are given. *G. Hochschild* (Urbana, Ill.).

Herz, Jean-Claude. Pseudo-algèbres de Lie. II. C. R. Acad. Sci. Paris **236**, 2289–2291 (1953).

In the notation of the preceding review, it is shown that every pseudo Lie algebra of dimension >1 over a non-commutative division ring K is obtained from a linear map φ of its underlying space into K by putting

$$[u, v] = \varphi(u)v - \varphi(v)u.$$

If K is commutative, those elements u of a pseudo Lie algebra E of dimension >1 for which $uK = (0)$ constitute an ideal N of E which contains all proper ideals of E and is an ordinary Lie algebra. The factor algebra E/N is isomorphic with a subalgebra of the pseudo Lie algebra of the derivations of K. *G. Hochschild* (Urbana, Ill.).

Shimura, Gorô. On a certain ideal of the center of a Frobeniusean algebra. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **2**, 117–124 (1952).

Let A be a Frobenius algebra over a field K. If $u_1, u_2, \cdots, u_n$ is a basis of A, there exists a second basis $v_1, v_2, \cdots, v_n$ such that the first regular representation of A formed with regard to this new basis coincides with the second regular representation of A formed with regard to the original basis. The set of all elements of the form $\sum_i u_i \alpha v_i$ with $\alpha \in A$ forms an ideal $\mathfrak{c}$ of the center Z of A which does not depend on the choice of the bases. The properties of this ideal are studied. In particular, it is shown that $\mathfrak{c} = Z$ if and only if A is separable (i.e., if A is semisimple and remains semisimple for every ground field extension). In general, $A\mathfrak{c}^2$ is the maximal separable ideal of A. If A is symmetric and if the irreducible representations of A in K are all absolutely irreducible, the rank of $\mathfrak{c}$ over K is equal to the rank modulo the characteristic of K of the matrix of Cartan invariants of A. Some extensions of the results to the case of quasi-Frobenius algebras are indicated. The case of group algebras is studied further. *R. Brauer* (Cambridge, Mass.).

Ikeda, Masatoshi. On a theorem of Gaschütz. Osaka Math. J. **5**, 53–58 (1953).

An algebra A over a commutative ring R with unity element 1 is said to be a Frobenius algebra if 1 is also the unity element of A and if the right and left regular representations of A are equivalent. Two bases of A are said to be dual if the left regular representation relative to the first is the same as the right regular representation relative to the second. Let P be a ring whose center contains R. A module $\mathfrak{M}$ is called an A-P-module if $\mathfrak{M}$ is a left A-module as well as a right P-module and satisfies the conditions $(a\omega)m = (am)\omega$, $(am)\rho = a(m\rho)$ for every a in A, m in $\mathfrak{M}$, ω in R, and ρ in P.

An A-P-module is said to be an M_μ-module if $1 \cdot m = m$ for all m in $\mathfrak{M}$ and if for every A-P-module $\mathfrak{N}$ containing $\mathfrak{M}$ a direct decomposition $\mathfrak{N} = \mathfrak{M} + \mathfrak{M}'$ as a P-module implies a direct decomposition $\mathfrak{N} = \mathfrak{M} + \mathfrak{M}''$ as an A-P-module. An M_0-module is defined analogously by the requirement that if $\mathfrak{N}/\mathfrak{N}' = \mathfrak{M}$ then a direct decomposition $\mathfrak{N} = \mathfrak{N}' + \mathfrak{M}'$ as a P-module implies a direct decomposition $\mathfrak{N} = \mathfrak{N}' + \mathfrak{M}''$ as an A-P-module. The main result of this paper is a proof that an A-P-module $\mathfrak{M}$ is an M_μ- or M_0-module if and only if there exists a P-endomorphism β of $\mathfrak{M}$ such that $\sum_i u_i \beta v_i$ is the identity endomorphism of $\mathfrak{M}$ for every R-basis $\{u_i\}$ of A and its dual basis $\{v_i\}$. This is a generalization of a theorem of W. Gaschütz [Math. Z. **56**, 376–387 (1952); these Rev. **14**, 533] which is in turn a generalization of the theorem of full reducibility of representations of a finite group over a field whose characteristic does not divide the order of the group. *R. M. Thrall* (Ann Arbor, Mich.).

Lenz, Hanfried. Über endliche Automorphismengruppen unendlicher Körpererweiterungen. Arch. Math. **4**, 100–106 (1953).

The author uses methods of Artin, Schreier, and Krull [Artin, Abh. Math. Sem. Univ. Hamburg **3**, 319–323 (1924); Artin and Schreier, ibid. **5**, 225–231 (1927); Krull, Math. Ann. **100**, 687–698 (1928)], considers infinite cyclotomic extensions, composites of purely infinite ideal cyclic extensions, extensions closed under certain kinds of root-extractions, etc., and proves theorems on nonexistence of automorphisms with period divisible by certain primes.

G. Whaples (Bloomington, Ind.).

Kneser, Martin. Die Norm einer Algebra. Arch. Math. **4**, 97–99 (1953).

Using the theory of factor sets over Galois-algebras [for definition see Knobloch, Math. Nachr. **6**, 11–20 (1951); these Rev. **13**, 314], the author proves Theorem 1: If K and K' are Galois algebras over k with the automorphism groups $\mathfrak{G}$ and $\mathfrak{G}'$, if l is a finite extension field of k, if $L=K\times_k l$ and $L'=K'\times_k l$, and if the crossed products $(L/l, \mathfrak{G}, \mathfrak{a})$ and $(L'/l, \mathfrak{G}', \mathfrak{a}')$ are similar over l, then $(K/k, \mathfrak{G}', N_{L/K}\mathfrak{a})$ and $(K'/k, \mathfrak{G}', N_{L'/K'}\mathfrak{a}')$ are similar over k. Here k is an arbitrary field and the norms are defined by taking determinants of a matrix representation. The author uses this theorem to give a new and simpler proof of a theorem due to the reviewer [Ann. of Math. (2) **55**, 367–372 (1952); these Rev. **13**, 720], incidentally removing the reviewer's assumption that l/k is separable. (R. MacKenzie [unpublished] had already shown that the reviewer's proof could be modified so as to remove this assumption). Using results of Jehne on Galois algebras [Abh. Math. Sem. Univ. Hamburg **18**, 70–98 (1952); these Rev. **14**, 729], the author then shows that if l/k is separable then every algebra class over l contains crossed products of the special form mentioned in Theorem 1, thus proving Theorem 2: The mapping of Theorem 1 gives a homomorphism of the group of all algebra classes over l into the group of algebra classes over k. It is mentioned that this mapping is the same as the transfer (Verlagerung) mapping used by Witt [J. Reine Angew. Math. **190**, 231–245 (1952); these Rev. **14**, 845]. Artin and Tate [unpublished] had previously shown that the reviewer's theorem was a consequence of such a more general homomorphism. *G. Whaples.*

***Delone, B. N. Asymptotic formulas in Galois theory.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 767–770. Akadémiai Kiadó, Budapest, 1952. (Russian and Hungarian)

For given field k and finite group G the author considers three problems. 1. Is there a Galois extension K of k with Galois group G? 2. Give a method of constructing all such K. 3. Find the "density" in one sense or another of those polynomials in $k[X]$ with Galois group G. After recalling some cases in which answers are known, he mentions some asymptotic formulae of more recent vintage, due to himself and to Faddeev, which answer, to a certain extent, some cases of problem 3. *E. R. Kolchin* (New York, N. Y.).

Kowalsky, Hans-Joachim. Zur topologischen Kennzeichnung von Körpern. Math. Nachr. **9**, 261–268 (1953).

The author gives a fresh exposition of the theory of locally compact division rings, using as his basic tool fields of type V. He discusses completely the case of characteristic 0, but in the case of characteristic p does not obtain the finite dimensionality over the center. *I. Kaplansky.*

Theory of Groups

Sánchez-Díaz, Rafael. Definitions of group involving quasi-inverse elements. Proc. Amer. Math. Soc. **4**, 424–428 (1953).

If Γ is any groupoid and $a \in \Gamma$, an element a' of Γ is a right quasi-inverse of a if $(ba)a'=b$ for every element b of Γ. Left quasi-inverses are defined similarly. Three independent sets of postulates for groups and four independent sets for Abelian groups are given in terms of quasi-inverse elements instead of in terms of a unit and inverse elements. Since quasi-inverses are defined without reference to a unit, the existence of a unit is not included in any of the postulate sets given. *D. C. Murdoch* (Vancouver, B. C.).

Douglas, Jesse. On the basis theorem for finite abelian groups. IV. Proc. Nat. Acad. Sci. U. S. A. **39**, 307–310 (1953).

[For parts I–III see same Proc. **37**, 359–362, 525–528, 611–614 (1951); these Rev. **13**, 203, 431.] By an induction on the exponent of the power of the prime p, the author proves the Basis Theorem for finite abelian groups by proving it for the p-primary components of these groups.

F. Haimo (St. Louis, Mo.).

Fuchs, L. A simple proof of the basis theorem for finite abelian groups. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 117–118 (1953).

The author proves the Basis Theorem for finite abelian groups by considering, for such a group A, a minimal generating system S: $(a_1, a_2, \cdots, a_k)$ such that (i) each member of S is of prime power order, and (ii) no element of S can be replaced by an element of smaller order in such a way that the modified set S' is a set of generators of A. A result of this method is that any abelian torsion group with a minimal set S of generators satisfying (ii) is a direct sum of cyclic groups. *F. Haimo* (St. Louis, Mo.).

Mann, H. B. An addition theorem of Abelian groups for sets of elements. Proc. Amer. Math. Soc. **4**, 423 (1953).

Let G be a finite abelian group, and A, B sets of elements in G. Let (A) denote the number of (distinct) elements in A; and let AB denote the set of all elements ab (a in A, b in B). The author proves that if A has the property that for every subgroup H of G either $(AH)\geq(A)+(H)-1$ or $AH=G$, then A has the same property when H is replaced by any subset B of G. The proof depends on both theorems of the author's previous paper [Canadian J. Math. **4**, 64–66 (1952); these Rev. **13**, 720]. *H. Davenport* (London).

Schützenberger, Marcel Paul. Sur l'extension d'un groupe de permutations d'un ensemble fini à l'ensemble des parties de celui-ci. C. R. Acad. Sci. Paris **236**, 449–450 (1953).

A permutation group G on a finite set E induces permutations on the subsets of E and these fall into classes of subsets X_i taken into each other by G. There occur incidences between these subsets and we say X_i has the incidence number a_j^i with X_j if every set in X_j contains a_j^i sets of X_i. The power of X_i is the index h_i of the subgroup of G fixing a subset in X_i. If A is the matrix (a_j^i) and H the diagonal matrix of the h_i, then various properties of A and H are stated, for example that $A=\exp C$ where C is a matrix given by the a_j^{i+1}. It is said that similar results hold for subsets of a finite projective space under a subgroup of its collineation group. No proofs are given. *Marshall Hall, Jr.*

Nagata, Masayoshi. Note on groups with involutions. Proc. Japan Acad. **28**, 564–566 (1952).

A finite group G with an involution fixing only the identity is abelian and this involution is $g^\sigma = g^{-1}$ for every $g \in G$. Infinite groups with such an involution are not necessarily abelian, since, for example, the free group with two generators has such an involution which interchanges the generators. Certain added assumptions are sufficient to prove such a group abelian. It is shown that each of the following suffices: (1) every element is of finite order; (2) every element is a square and if $g^\sigma = g^{-1}$ with $g = h^2$, then $h^\sigma = h^{-1}$; (3) g and g^σ generate a nilpotent group. The second condition is very close to a result of B. H. Neumann [J. London Math. Soc. **15**, 203–208 (1940); these Rev. **2**, 121].

Marshall Hall, Jr. (Columbus, Ohio).

*Rédei, László, and Szép, Jenő. **Finite nilpotent groups.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 225–232. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian summary)

This paper has appeared in German in Monatsh. Math. **55**, 200–205 (1951); these Rev. **13**, 203.

Čarin, V. S. On the theory of locally nilpotent groups. Mat. Sbornik N.S. **29(71)**, 433–454 (1951). (Russian)

Let $f(x)$ be an irreducible, non-cyclotomic polynomial over the rationals. For a positive integer $s \geqq 2$, let $\nu(f_n)$ be the least of the orders of the irreducible factors of $f(x^{s^n})$. The author shows that $\lim_{n\to\infty} \nu(f_n) = \infty$. Let A be a non-singular matrix over the rationals with the property that the equations $X^n = A$ all have solutions ($n = 1, 2, 3, \cdots$). Then, by the result quoted above, it is proved that the eigenvalues of A are all 1. Let a complete group be one in which equations of the form $x^n = g$ are always solvable. A complete group of matrices over the rationals turns out to be aperiodic, nilpotent, and of finite rank. Conversely, groups with these three properties are precisely those which can be represented by triangular matrices over the rationals with unities along the main diagonal. In order for a group to be complete, nilpotent, and of finite rank, it is necessary and sufficient that the group possess a finite normal chain which sweeps out the group with complete, locally cyclic factors. The extension of a complete nilpotent group of finite rank by a complete, locally nilpotent group is a complete, locally nilpotent group with a non-trivial center. If a locally nilpotent aperiodic group $\mathfrak{G}$ has a normal subgroup of finite rank, then $\mathfrak{G}$ has a non-trivial center. A solvable group $\mathfrak{G}$ of finite rank with the wider completeness property (that the x^n for each positive integral n generate $\mathfrak{G}$) is nilpotent.

F. Haimo (St. Louis, Mo.).

Honda, Kin-ya. On finite groups, whose Sylow-groups are all cyclic. Comment. Math. Univ. St. Paul. **1**, 5–39 (1952).

Some of the results of this paper were announced without proof in an earlier abstract of the same title [Proc. Japan Acad. **25**, 154–159 (1949); these Rev. **14**, 445]. The present paper contains complete statements of the theorems and their proofs. Some of the additional results are now given. The commutator subgroup of a hypercyclic group is the direct product of some of the Sylow subgroups. The product of all the normal Sylow subgroups of a hypercyclic group is cyclic with cyclic factor group. There are 108 hypercyclic subgroups of order $\leqq 100$. A principal result is the Ramification Theorem: Let P be a finite p-group, and let G be a hypercyclic group. Suppose that there exist two semi-direct products of P by G with respective homomorphisms α_1 and α_2 of G into the automorphism group of P in such a way that the products are both hypercyclic groups with Sylow subgroup P. Suppose, further, that α_1 and α_2 have the same kernel F. The mappings α_1 and α_2 can be made to correspond to elements d_1 and d_2 of G. Let W be the centralizer of the commutator subgroup of G. Then the two semi-direct products are isomorphic if, and only if, d_1 and d_2 are congruent modulo WF.

F. Haimo (St. Louis, Mo.).

Baer, Reinhold. Group elements of prime power index. Trans. Amer. Math. Soc. **75**, 20–47 (1953).

The subject of this paper is sufficiently indicated by its title, as soon as one knows that the index of a group element is the number of its conjugates, and that all groups considered are finite. Its scope is given, roughly, by the following representative results.

First, a classical result of Burnside says that a group containing elements of prime power index (not 1) has a proper normal subgroup. Here normal can be improved to characteristic; and thence to fully invariant, by noting that a finite group G with no proper fully invariant subgroup has no proper characteristic subgroup either. For if A is any group, let $X(G, A)$ and $Y(G, A)$ be the sets of elements of G mapped onto 1 in every homomorphism of G onto A, and into A, respectively. $Y(G, A)$ is always a fully invariant subgroup of G. But if A is a homomorphic image of G of minimum order, then $X(G, A) = Y(G, A) \neq G$, so that $X(G, A) = 1$. Since A is simple, this implies that G is a direct product of a number of copies of A.

Next, suppose that in G every element of prime power order also has prime power index. This, the author shows, implies that G is a direct product of factors of coprime orders, each of which is either a p-group or a group with abelian Sylow subgroups whose order is divisible by just two primes. The elements whose order and index are powers of the same prime have a close connection with the hypercentre $H(G)$. Thus if P is a Sylow p-subgroup of G, an element of P belongs to $H(G)$ if and only if it belongs to a normal subgroup of P every element of which has index (in G) a power of p. It follows that $H(G)$ is the intersection of the normalisers of all Sylow subgroups of G, and is also the intersection of all maximal nilpotent subgroups of G.

The proofs throughout the paper are elementary, but they lean heavily on earlier results (particularly the theorem of Burnside quoted above) the proofs of which require much deeper methods.

G. Higman (Manchester).

Huppert, Bertram. Über das Produkt von paarweise vertauschbaren zyklischen Gruppen. Math. Z. **58**, 243–264 (1953).

Let G be a p-group of order p^n which is the permutable product of two finite cyclic groups. Then the Frattini subgroup Φ of G is generated by the pth powers of the elements of G, and both Φ and the center are products of pairs of cyclic groups. If p is odd, G/Φ has order p^d, $d \leqq 2$, G can be decomposed into the product of a pair of disjoint cyclic groups, G has a cyclic group as derivative and G has at most $p+1$ subgroups of order p. If $p = 2$, then G' ($\Phi(G')$ has order 2^d, $d \leqq 2$. If each maximal subgroup of such a G has order $\leqq 4$, then G has a cyclic normal subgroup with a cyclic factor group. In the second part of the paper, it is proved that the product of two finite hamiltonian or of two finite abelian groups is solvable. Let G be any finite group which is the

permutable product of two cyclic groups. Then the Sylow group corresponding to the largest prime divisor of the order of G is a normal subgroup of G. If G induces a factorization of a subgroup of G, it induces like factorizations into the normalizer and centralizer of this subgroup. At least one of the factors of G includes a non-trivial normal subgroup of G. There exists an ascending chain of normal subgroups without refinements with primes for the orders of the factor groups. If such a G is nilpotent, the derivatives of the various Sylow subgroups corresponding to a fixed prime coincide, and the direct product of these derivatives is G'. If G has odd order, G' is abelian, and G' is cyclic if all the Sylow subgroups (except possibly those corresponding to the smallest prime divisor of the order of G) are non-abelian. If G has more than two cyclic factors, the ascending chain theorem above still holds. If, in addition, G is a p-group, then the mutual commutator subgroup of the subgroups generated by the p^ith powers and by the p^jth powers of the elements of G is included in the subgroup generated by the p^{i+j}th powers of the elements of G' which in turn is included in the subgroup generated by the p^{i+j+1}st powers of the elements of G. Reference is made to H. Wielandt [Math. Z. 55, 1–7 (1951); these Rev. 13, 621]. *F. Haimo.*

Gol'berg, P. A. Sylow bases of infinite groups. Mat. Sbornik N.S. 32(74), 465–476 (1953). (Russian)

If G is a group, the set S: $S_1, S_2, \cdots$ of Sylow subgroups forms a Sylow basis if no element of $\{S_\alpha\}$, $\alpha \in T$, is divisible by any primes not corresponding to the subgroups involved, whenever T is a subset of the integers, and if $\{S\}$ is G. Theorem. Let S be a Sylow basis; let $N(S)$ denote the intersection of the normalisors of S and all bases conjugate to S in G. If the factor group $G/N(S)$ is locally finite and satisfies the descending chain condition for subgroups (d.c.c.), any two Sylow bases of G are mutually conjugate. This generalizes a result of Baer [Duke Math. J. 6, 598–614 (1940); these Rev. 2, 2] where $G/N(S)$ had to be finite. The paper shows that the theory of Sylow bases and Sylow π-bases (condition restricted to Sylow subgroups corresponding to a set π of primes) can be extended from finite groups to locally finite groups with d.c.c. [The reviewer can supply a translation of this paper.] *J. L. Brenner.*

Kemhadze, Š. S. Uniqueness bases in infinite regular p-groups. Ukrain. Mat. Žurnal 4, 57–64 (1952). (Russian)

The author considers infinite groups G in which any two elements generate a finite regular p-group. Let G have an L-series (presumably transfinite)

$$1=L_0\subset\cdots L_i\subset L_{i+1}\subset\cdots\subset L_\tau=G$$

in which each L_i is a normal subgroup and in which each factor group L_{i+1}/L_i is cyclic. If also the orders of elements of G are bounded, then G has a uniqueness basis.

Marshall Hall, Jr. (Columbus, Ohio).

Nagao, Hiroshi. A note on extensions of groups. Proc. Japan Acad. 25, no. 10, 11–14 (1949).

Any extension G of a group N by a group A is determined by a factor set and a homomorphism σ of A into the residue class group of the automorphism group of N by its inner automorphism group. Let Z be the center of N. The author shows that if G is any second extension of N by A with the same σ then G'/Z is isomorphic to G/Z. This result essentially reduces the problem of determining all extensions of N by A to the case when N is abelian. An extension G of N by A is said to be central if N lies in the center of G. The group of central extensions of N by A is determined in the case in which N and A are abelian and A has a finite number of generators. *R. M. Thrall* (Ann Arbor, Mich.).

Clowes, J. S., and Hirsch, K. A. Simple groups of infinite matrices. Math. Z. 58, 1–3 (1953).

Let $\mathfrak{F}$ be a fixed field. Consider all infinite row-finite matrices over $\mathfrak{F}$ which have arbitrary finite unimodular upper left-hand corners and zeros elsewhere except for unity at every place along the remainder of the main diagonal. It is proved that the group of all such matrices is simple. Reference is, especially, to L. E. Dickson [Linear groups, Teubner, Leipzig, 1901, pp. 84–85]. *F. Haimo.*

Hall, Marshall, Jr. Subgroups of free products. Pacific J. Math. 3, 115–120 (1953).

According to a theorem of Kurosh [Math. Ann. 109, 647–660 (1934)] a subgroup $\neq 1$ of a free product G of groups A_α is itself a free product of a free group and of conjugates of subgroups of free factors A_α of G. In this paper the author gives a new proof of purely algebraic nature of this theorem which is shorter and simpler than the proof so far known. *T. Szele* (Debrecen).

Gruenberg, K. W. Two theorems on Engel groups. Proc. Cambridge Philos. Soc. 49, 377–380 (1953).

A group G is said to satisfy the Engel condition if all simple commutators $(x, y, \cdots, y)$ with a sufficient number of y's are equal to the identity. If a fixed number n of y's suffices in every case, then we say that G satisfies the nth Engel condition. It is shown here that every finitely generated solvable group satisfying the Engel condition is nilpotent. The proof depends on Philip Hall's commutator collecting process. *Marshall Hall, Jr.* (Columbus, Ohio).

Hamill, Christine M. A collineation group of order $2^{13}\cdot 3^5\cdot 5^3\cdot 7$. Proc. London Math. Soc. (3) 3, 54–79 (1 plate) (1953).

Following the methods of Todd [Proc. Roy. Soc. London. Ser. A. 189, 326–358 (1947); these Rev. 9, 6] the author studies a family of three finite primitive collineation groups $\mathcal{G}^5$, $\mathcal{G}^6$, $\mathcal{G}^7$ which contain homologies of period two, and determines the conjugacy classes by a study of certain configurations in projective 7-space obtained from the vertices of the polytope $(PA)_8$. These groups are isomorphic with the rotation groups of the semi-regular polytopes $(IA)_8$, $(SA)_7$, and $(PA)_8$. The classifications of $\mathcal{G}^5$ and $\mathcal{G}^6$ into conjugacy classes agree with those obtained in another way by Frame [Ann. Mat. Pura Appl. (4) 32, 83–119 (1951); these Rev. 13, 817]. The large group $\mathcal{G}^7$ of order 348,364,800 which contains $\mathcal{G}^6$ as a subgroup of index 240, is analyzed into its 67 classes. *J. S. Frame* (East Lansing, Mich.).

Shephard, G. C. Unitary groups generated by reflections. Canadian J. Math. 5, 364–383 (1953).

A p-fold reflection is a unitary transformation of period p that leaves a hyperplane invariant. Groups generated by n such reflections of orders p_i $(i=1, 2, \cdots, n)$ in hyperplanes of E_n may be described by a graph of n nodes in which (1) the ith node is labelled p_i if $p_i>2$; (2) the ith and jth nodes are connected by a link if their hyperplanes are not orthogonal; (3) the link is labelled $k/2$ if the two reflections generate a subgroup of order $k>6$; and (4) a circuit of three or more connected nodes is characterized by an integer m. The author studies a class of groups $[p\ q; r]$

generated by $p+q+r$ 2-fold reflections, which for $m=2$ includes the real orthogonal groups $[3^{p,q,r-1}]$ studied by Coxeter [Regular polytopes, Methuen, London, 1948; these Rev. 10, 261].

Besides the true unitary type $[1\,1;n-2]^m$ of order $m^{n-1}\cdot n!$, found for all m, there are for $m>2$ just six special groups of which $[2\,1;2]^3$, $[2\,1;3]^3$ and $[2\,1;1]^4$ have the finite orders $72\cdot 6!$, $108\cdot 9!$, and $64\cdot 5!$ respectively, and three others are infinite. Abstract definitions for all these groups are given, followed by a description of the associated polytopes. *J. S. Frame* (East Lansing, Mich.).

Nagao, Hirosi. Note on the modular representations of symmetric groups. Canadian J. Math. 5, 356–363 (1953).

Let $\mathfrak{A}=\mathfrak{A}(n,p)$ denote the group algebra of the symmetric group of degree n over a field of characteristic p. A two-sided ideal which is a minimal direct component of $\mathfrak{A}$ is called a block. Each block is associated with integers m and β for which $n=m+\beta p$. The numbers of ordinary and of modular irreducible representations in a block depend only on the value of β. The main result in the present paper is a new direct computation for the number of modular irreducible representations in a given block. The method used is to first show that this number can be calculated in terms of the dimension of a certain module, and then the dimension of the module is determined. *R. M. Thrall.*

Murnaghan, F. D. The parametrisation and element of volume of the unitary symplectic group. Proc. Nat. Acad. Sci. U. S. A. 39, 324–327 (1953).

The author gives a system of parameters for the $2k$-dimensional unitary symplectic group and a formula for the element of volume in the group manifold including the in-class factor as well as the well-known class factor.

If E_k denotes the k-rowed unit matrix, the unitary group is considered which leaves invariant the matrix

$$\begin{bmatrix} 0 & -E_k \\ E_k & 0 \end{bmatrix}.$$

Any matrix of the group may be transformed to the canonical form

$$\begin{bmatrix} D & 0 \\ 0 & \bar{D} \end{bmatrix}$$

where $D=\operatorname{diag}[e^{i\alpha_1}, e^{i\alpha_2}, \cdots, e^{i\alpha_k}]$. The class factor for the element of volume is well known to be

$$\sin^2\alpha_1 \sin^2\alpha_2 \cdots \sin^2\alpha_k \prod\{1-\cos(\alpha_p-\alpha_q)\} \times\{1-\cos(\alpha_p+\alpha_q)\}\,d\alpha_1\cdots d\alpha_k.$$

To obtain the $2k^2$ in-class parameters it is observed that any matrix of the group may be transformed into the canonical form by a sequence of k^2 transforming matrices each involving two parameters. Thus, for $k=3$ two such transforming matrices are denoted by

$$(23)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & ce^{i\tau} & -s & 0 & 0 & 0 \\ 0 & s & ce^{-i\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ce^{-i\tau} & -s \\ 0 & 0 & 0 & 0 & s & ce^{i\tau} \end{bmatrix},$$

$$(14)=\begin{bmatrix} ce^{i\tau} & 0 & 0 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ s & 0 & 0 & ce^{-i\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $c=\cos\beta$, $s=\sin\beta$. If β, τ are replaced by β_i, τ_i, the corresponding symbols are replaced by $(23)_i$, $(14)_i$.

For $k=1$ the transforming matrix is (12); for $k=2$ the sequence of four transforming matrices is $(12)_4\,(24)_3\,(12)_2\,(13)_1$; and for $k=3$ the set of 9 is $(12)_9\,(23)_8\,(36)_7\,(12)_6\,(23)_5\,(12)_4\,(25)_3\,(12)_2\,(14)_1$. The general form is apparent from these examples.

The in-class factor for the element of volume for $k=1$ is

$$dV_2=\sin 2\beta\, d\beta\, d\tau;$$

for $k=2$,

$$dV_4=(\cos^4\beta_2)(\sin^2\beta_2)\prod(\sin 2\beta)\,(d\beta)\,(d\tau);$$

for $k=3$,

$$dV_6=(\cos^6\beta_3)(\cos^4\beta_2)(\sin^2\beta_3\sin^4\beta_7\sin^2\beta_8)(\sin^2\beta_3) \times\prod(\sin 2\beta)\,(d\beta)\,(d\tau).$$

Some indication is given as to the form of these factors in the general case, but the description is not quite explicit. *D. E. Littlewood* (Bangor).

Inonu, E., and Wigner, E. P. On the contraction of groups and their representations. Proc. Nat. Acad. Sci. U. S. A. 39, 510–524 (1953).

The authors examine in various special cases relationships between the representations of a Lie algebra and those of a limit of the Lie algebra [concerning the latter notion, see Segal, Duke Math. J. 18, 221–265 (1951); these Rev. 13, 534]. The basic case noted is that in which the limit Lie algebra is obtained from the original one by a singular linear transformation. The most general such limit has the form $M\oplus N'$ (Lie direct sum), where M is a subalgebra of the original Lie algebra and N' is an abelian Lie algebra, and is called a contraction of the original algebra. Contraction of the inhomogeneous Lorentz group yields the full Galilean group, and the authors trace the lack of physical significance of the unitary representations of this group to the circumstance that they are in a certain way limits of unitary representations of the inhomogeneous Lorentz group of imaginary rest mass. Other contractions considered in relation to the unitary representations of the groups involved are those of the 3-dimensional rotation group to the euclidean group in 2 dimensions and the affine group in 1 dimension to an abelian group (one irreducible unitary representation of the affine group is overlooked in this treatment). *I. E. Segal* (Chicago, Ill.).

Metz, André. Les transformations de Lorentz ne forment pas, en général, un "groupe". Signification physique de cette propriété. C. R. Acad. Sci. Paris 237, 29–31 (1953).

Elementary considerations on the successive application of rotations and Lorentz transformations. *H. P. Robertson* (Pasadena, Calif.).

Iwasawa, Kenkichi. Hilbert's fifth problem. Sûgaku 1, 161–171 (1948). (Japanese)

This is a summary of an invited address given at a meeting of the Mathematical Society of Japan in October 1947. The author first proves a result of C. Chevalley and A. Malcev that connected locally Euclidean solvable groups are Lie groups. Then the notion of (L)-groups is introduced and various properties of (L)-groups are discussed. Most of the

results of this address were published in a subsequent paper (in English) of the author [Ann. of Math. (2) **50**, 507–558 (1949); these Rev. **10**, 679]. The address is concluded with the conjectures that (i) all connected locally compact groups are (L)-groups and (ii) all connected locally compact groups having no arbitrarily small subgroups or invariant subgroups are Lie groups. *S. Kakutani.*

Dieudonné, Jean. Sur les groupes de Lie algébriques sur un corps de caractéristique $p>0$. Rend. Circ. Mat. Palermo (2) **1** (1952), 380–402 (1953).

Let V be a finite-dimensional vector space over an algebraically closed field K of characteristic $p\neq 0$. Let G be an algebraic group of automorphisms of V, and let g denote the Lie algebra of G, in the sense of Chevalley [Théorie des groupes de Lie, vol. II, Hermann, Paris, 1951; these Rev. **14**, 448]. The present paper is concerned with the correspondence between algebraic subgroups of G and subalgebras of g which, over fields of characteristic $p\neq 0$, is still quite problematic and must necessarily remain defective, in view of examples given by Chevalley.

It is observed first that g must be closed under the operation $X\to X^p$, so that, with every endomorphism X of V, g must also contain the algebra $P(X)$ which is spanned by the p-powers X^{p^k}. It is then shown that, if S and N are the semisimple and nilpotent components of X, $P(S)$ has a basis consisting of semisimple endomorphisms of V whose characteristic roots lie in the prime field of K, and which, together with the non-zero p-powers of N constitute a basis for $P(X)$. The further results concern the following two typical extreme cases.

The first case is that of an irreducible algebraic group G of diagonal matrices. By first determining the structure of G in an explicit form and then using the above result concerning $P(S)$ it is shown that if h is any subalgebra of g which is closed also under the operation $X\to X^p$, there exists an irreducible algebraic subgroup H of G whose Lie algebra is h.

The second case is that of the irreducible algebraic group J consisting of all endomorphisms

$$s=I+u_1N+\cdots+u_{e-1}N^{e-1},$$

where $u_i\in K$ and N is a nilpotent endomorphism of exponent e. By an intense study of the exponential series, the author obtains a decomposition of J into 'elementary groups' $J_k(N)=J_1(N^k)$, where k ranges over the positive integers less than e and prime to p. Here $J_1(N)$ is an irreducible algebraic group of dimension $r+1$, where r is the largest exponent for which p^r is less than e, and the Lie algebra of $J_1(N)$ is $P(N)$. The elements of $J_1(N)$ are the polynomials in N which are obtained by expanding the exponential series $\exp(u_0N+u_1N^p+\cdots+u_rN^{p^r})$. In fact, it is shown that the coefficients of the non-vanishing powers of N which occur here can be written with denominators prime to p, so that one obtains a meaningful expression on reducing them mod p. A very elaborate further analysis of J finally shows that every irreducible algebraic subgroup G of J whose Lie algebra contains N has an irreducible algebraic subgroup H of (the minimal) dimension $r+1$ whose Lie algebra still contains N. *G. Hochschild.*

Dobrescu, Andrei. On the classification of Lie groups with four parameters. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 137–146 (1950). (Romanian. Russian and French summaries)

By use of the vector and tensor of structure [cf. Vranceanu, Leçons de géométrie différentielle, vol. 1, Bucharest, 1947, pp. 105–111; these Rev. **9**, 532] the author succeeds in classifying all four-parameter Lie groups with non-vanishing structure vector. In all, nine distinct canonical structures result. *J. L. Vanderslice.*

NUMBER THEORY

Nanda, V. S. Tables of solid partitions. Proc. Nat. Inst. Sci. India **19**, 313–314 (1953).

In a previous paper [Proc. Cambridge Philos. Soc. **47**, 591–601 (1951); these Rev. **13**, 895] the author considered plane and solid partitions. He denotes by $p^{(3)}(n,m)$ the number of solid partitions of n in which the smallest summand is m. Accordingly

$$p^{(3)}(n,n)=n(n+1)/2$$
$$p^{(3)}(n,m)=0 \quad \text{for} \quad n/2<m<n.$$

Hence the tabulation of this function is confined to $m\leqq n/2$. The table is for $n=2(1)25$. The entries $p^{(3)}(n,1)$ give the total number of unrestricted solid partitions of $n-1$. This function is generated by

$$\prod_{r=1}^{\infty}(1-x^r)^{-r(r+1)/2}=1+x+4x^2+10x^3+\cdots.$$

D. H. Lehmer (Berkeley, Calif.).

Goldberg, Karl. A table of Wilson quotients and the third Wilson prime. J. London Math. Soc. **28**, 252–256 (1953).

Wilson's quotient is defined as the integer

$$w_p=[(p-1)!+1]/p,$$

where p is a prime. The table gives the least positive residue of w_p modulo p for $p<10000$. The interesting case of $p=563$ is the only one except $p=5$ and $p=13$ for which w_p is divisible by p. Wilson's quotient appears in the arithmetic theory of Bernoulli numbers, Fermat's Last Theorem and congruence properties of symmetric functions. The calculation of the table was performed on the SEAC.

D. H. Lehmer (Berkeley, Calif.).

***Vorob'ëv, N. N. Čisla Fibonačči. [Fibonacci numbers.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 48 pp. .75 rubles.

No. 6 in the series, "Popular lectures on mathematics."

Knödel, Walter. Carmichaelsche Zahlen. Math. Nachr. **9**, 343–350 (1953).

The term Carmichael number is used to denote a composite number n for which the congruence

$$a^{n-1}\equiv 1 \pmod n$$

holds for all a prime to n. These numbers have been the subject of numerous investigations. The author considers more generally "C_k-numbers" defined as integers n for which the congruence

$$a^{n-k}\equiv 1 \pmod n$$

holds for all a prime to n. Here $n>k>0$. The following results are proved. Let

$$n=2^{n_2}3^{n_3}\cdots, \quad k=2^{k_2}3^{k_3}\cdots$$

be the decomposition of n and k into powers of primes, then

$$\begin{aligned} n_i &\leqq k_i+1 \quad \text{if} \quad i>0 \\ n_0 &\leqq k_0+2 \quad \text{if} \quad k_0>0 \\ n_0 &= k_0 \quad \text{if} \quad k_0=0. \end{aligned}$$

In particular, if $(n, k)=1$, then n is an odd squarefree number. All sufficiently large n are of the form $n=mpq$ with $p<3m^2$, $q<4m^2$, $p<q$, p and q primes. The number of C_k-numbers less than x is $O(k^6+2^{-t}kx)$, where $t=(\log x)^{1/2}$. It is conjectured that this is really $O(kx^\theta)$ for some $\theta<1$.

D. H. Lehmer (Berkeley, Calif.).

Hall, James S. A remark on the primeness of Mersenne numbers. J. London Math. Soc. **28**, 285–287 (1953).

The author suggests a modification of the Lucas test for the primality of Mersenne numbers $M_p=2^p-1$. He deplores the rapidity with which the sequence

$$4, 14, 194, 37634, \cdots, u_{k+1}=u_k^2-2$$

increases and suggests that the sequence be replaced by

$$3, 147, 1383123, 489735485064147, \cdots, h_{k+1}=h_k+(2^{k-1}h_k)^2, \quad h_3=3$$

with the condition that $h_{p-3}+8$ be divisible by M_p. The suggestion has no practical value, however, inasmuch as the terms of either sequence must be taken modulo M_p and the concept of rapidity of increase disappears. The more elaborate recurrence for the h's would appear to be the only drawback of the new test.

D. H. Lehmer.

Touchard, Jacques. On prime numbers and perfect numbers. Scripta Math. **19**, 35–39 (1953).

Let $\sigma(k)$ denote the sum of the divisors of k. The author considers the sum

$$S_p=\sum_{k=1}^{n-1}k^p\sigma(k)\sigma(n-k)$$

and derives the identities

$$(1) \qquad n^2(n-1)\sigma(n)=18n^2S_0-60S_2,$$

$$(2) \qquad n^2(n-1)\sigma(n)=48nS_2-72S_3$$

from van der Pol's differential equation

$$2y'''+2yy''-3y^2=0$$

for the function $y-1-24\sum_{k=1}^{n}\sigma(k)e^{-kt}$. The assumptions that n is a prime ($\sigma(n)=1+n$) or is perfect ($\sigma(n)=2n$) are made in (1) and (2) and certain consequences are derived. For example, odd perfect numbers must be of one of the forms $36m+1$, 9, 13, 25. The identities (1) and (2) follow also from certain formulas of Ramanujan. S_p is tabulated for $n=2(1)11$ and $p=0(1)3$.

D. H. Lehmer.

Wright, E. M. The calculation of large primes. Math. Gaz. **37**, 104–106 (1953).

The author points out that the method used by Miller, Wheeler and the EDSAC [cf. Miller, Eureka **1951**, no. 14, 10–11; these Rev. **13**, 436] to show that $180(2^{127}-1)^2+1$ is a prime can be extended to the case of numbers of the form kp^2+1. The following theorem is proved. Let n be an odd number of the form kp^2+1 where p is an odd prime and k is not a cube. If now $k<(8p+8)^{1/3}-2$, $2^k\not\equiv 1 \pmod n$, $2^{n-1}\equiv 1 \pmod n$, then n is a prime.

D. H. Lehmer.

Hall, Marshall, Jr. Some equations $y^2=x^3-k$ without integer solutions. J. London Math. Soc. **28**, 379–383 (1953).

In this paper the author gives two new methods for obtaining equations of the type mentioned in the title. In the first method he uses factorisation in $K(\sqrt{-k})$ in certain cases when $h(\sqrt{-k})\equiv 0 \pmod 3$. In the second, use is made of a representation $k=3a^2+rb^3$, applying cubic reciprocity to the equation in the form $x^3-rb^3=y^2+3a^2$. Three theorems are proved, and applications of these theorems are given. The equation is impossible for $k=24$, 31, and 77. The same principles can be used to determine other insoluble equations ($k=92$, -60).

W. Ljunggren (Bergen).

***Bronkhorst, Pieter. Over het aantal oplossingen van het stelsel diophantische vergelijkingen:**

$$\left.\begin{aligned} x_1^2+x_2^2+\cdots x_s^2&=n \\ x_1+x_2+\cdots x_s&=m \end{aligned}\right\} \text{ voor } s=6 \text{ en } s=8.$$

[On the number of solutions of the system of Diophantine equations:

$$\left.\begin{aligned} x_1^2+x_2^2+\cdots x_s^2&=n \\ x_1+x_2+\cdots x_s&=m \end{aligned}\right\} \text{ for } s=6 \text{ and } s=8.]$$

Thesis, University of Groningen, 1943. North Holland Publishing Co., Amsterdam, 1943. i+68 pp.

Let s be a fixed positive integer; let $n>0$ and m be variable integers, and let $r(n, m)$ denote the number of integral solutions of

$$x_1^2+x_2^2+\cdots+x_s^2=n, \quad x_1+x_2+\cdots+x_s=m.$$

The function $r(n, m)$ has been investigated by H. D. Kloosterman [Math. Ann. **118**, 319–364 (1942); these Rev. **5**, 33; **9**, 735] for odd squarefree s, and he gave explicit expressions for $r(n, m)$ when $s=3$, 5, or 7. In this thesis, the author obtains such explicit equations in the even cases $s=6$ and $s=8$.

K. Mahler (Manchester).

Rivier, William. A propos de la résolution en nombres entiers de l'équation à coefficients entiers $rx+sy=m$. Bull. Sci. Math. (2) **77**, 51–55 (1953).

Let $r>0$, $s>0$, $k\geqq 0$ be fixed integers with $(r, s)=1$, and let I_k be the interval $(krs, krs+rs-1)$. For m in the interval I_k the equation $rx+sy=m$ has either k or $k+1$ solutions, there being exactly k solutions for $(r-1)(s-1)/2$ values of m in I_k.

I. Niven (Eugene, Ore.).

Georgiev, G. On the solution in rational numbers of the indeterminate equation $\sum_{k=1}^{n}A_k\prod_{i=1}^{n}x_i^{a_{ki}}=A_0$. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 115–118 (1953). (Russian)

The title equation may be solved by a transformation of the type $x_i=\prod_{r=1}^{n}X_r^{\lambda_{ri}}$ if the integers λ_{ri} can be chosen so that $\det(\lambda_{ri})=\pm 1$ and so that $\sum_i \lambda_{ri}a_{ki}$ is 0 or 1 for each k, with at least one 1. For these the equation becomes a linear equation in X_n; and the x_i are uniquely determined by the X_i.

J. W. Cassels (Cambridge, England).

Davenport, H., and Erdös, P. The distribution of quadratic and higher residues. Publ. Math. Debrecen **2**, 252–265 (1952).

Some problems concerning the distribution of kth power residues and nonresidues are discussed. Let d denote the least quadratic nonresidue to the prime p. Then the authors show $d=O(p^{\beta}\log^{\beta} p)$, where $\beta=e^{-1/2}$, which result is slightly better than a result of Vinogradov. The method is based on an elementary lemma on characters, which lemma, however, as the authors remark in a note added later, is already given by Vinogradov [Foundations of the theory of numbers, 5th ed., Gostehizdat, Moscow-Leningrad, 1949, p. 109; these Rev. **12**, 10]. Similar results are deduced for $k>2$, which

for $k \geqq 4$ are more precise than those known until now. Further, the authors give for $k \geqq 3$ an estimate for the order of magnitude of the least kth power nonresidue in any given one of the $k-1$ classes of nonresidues. They show that a positive number $\eta = \eta(k)$ (depending on k only) exists, such that the estimate $O(p^{1-\eta})$ holds. In the case $k=3$ their result takes the form $O(p^{\gamma+\epsilon})$, where $\gamma = 1/2u = 0.383$ approximately, u denoting the solution of the equation

$$\log u + \int_1^{2u-1} \frac{\log t}{t+1}\,dt = \frac{1}{3},$$

and ϵ being an arbitrary positive constant. Finally the distribution of the quadratic residues and nonresidues in sets of consecutive integers is considered. *J. F. Koksma.*

Brauer, Alfred. On the distribution of the Jacobian symbols. Math. Z. 58, 226–231 (1953).

For m odd define M^{++} as the number of pairs $c, c+1$ in a complete residue system (mod m) such that the Jacobi symbols $(c/m) = (c+1/m) = +1$; the numbers $M^{+-}, \cdots, M^{00}$ are defined in a similar manner. Also let $\phi(m)$ denote the Euler ϕ-function and

$$\phi_2(m) = m \prod_{p \mid m} \left(1 - \frac{2}{p}\right)$$

the Schemmel function. It is shown that for m squarefree the M's can be evaluated in terms of ϕ and ϕ_2. For example, $M^{++} = M^{+-} = M^{-+} = M^{--} = \frac{1}{4}(\phi_2(m)+1)$ if m is divisible by an odd number of primes of the form $4k+1$. Also

$$M^{00} = m - 2\phi(m) + \phi_2(m);$$

this result holds for arbitrary m. *L. Carlitz.*

Carlitz, Leonard. A theorem of Glaisher. Canadian J. Math. 5, 306–316 (1953).

Let B_m denote the mth Bernoulli number in the notation of Nörlund. Put

$$(x-1)(x-2)\cdots(x-p+1) = x^{p-1} - A_1 x^{p-2} + \cdots + A_{p-1}.$$

In this paper the author first shows that if p is a prime such that $1 < r < \frac{1}{2}(p-1)$, then

$$2rA_{2r+1} \equiv -\tfrac{1}{2}p^2(p-2r-1)B_{2r} - (2r+1)p^2 \sum_{i=1}^{r-1} \frac{1}{4i} B_{2i}B_{2r-2i} \pmod{p^4}.$$

A similar but slightly more complicated congruence (mod p^5) is also obtained. From this result the author systematically develops numerous applications in the form of weaker congruences. In particular, the following theorem of Glaisher [Quart. J. Pure Appl. Math. 31, 321–353 (1900)] is deduced: $A_{2r}/p \equiv -B_{2r}/2r \pmod p$, $A_{2r+1}/p^2 \equiv (2r+1)B_{2r}/4r \pmod p$. Another typical application is the congruence

$$(-1)^m \binom{2m}{m} \equiv 2^{4m}\left(1 + \tfrac{1}{12}p^2 B_{p-2}\right) \pmod{p^4}.$$

In the remainder of the paper the author extends Glaisher's theorem to more general sequences. The generalization depends on the fact that the A_m can be expressed in terms of Bernoulli numbers of higher order, namely $A_r = (-1)^r \binom{p-1}{r} B_r^{(p)}$. Hence if $f(x) = \sum_{m=1}^{\infty} c_m x^m/m!$ $(c_1 = 1)$, where the c_m are integral (mod p), and we define $\beta_m^{(k)}$ by means of $(x/f(x))^k = \sum_{m=0}^{\infty} \beta_m^{(k)} x^m/m!$, it is natural to seek congruences satisfied by $\beta_m^{(p)}$. Further generalizations are obtained by introducing coefficients $\eta_m^{(k)}$ defined by $(1+af(x))^{-k} = \sum_{m=0}^{\infty} \eta_m^{(k)} x^m/m!$, where a is integral (mod p). *A. L. Whiteman* (Princeton, N. J.).

Carlitz, L. Some congruences for Bernoulli numbers of higher order. Quart. J. Math. Oxford Ser. (2) 4, 112–116 (1953).

Let the Bernoulli numbers of order k be defined by means of $(x/(e^x-1))^k = \sum_{m=0}^{\infty} B_m^{(k)} x^m/m!$ $(|x| < 2\pi)$. In this paper the author proves two theorems. If p is a prime not less than 5, then: (1) $B_{p-1}^{(p+1)} \equiv p^2/6 \pmod{p^4}$; (2) $B_p^{(p)} \equiv -p^2(p-1)!/2 \pmod{p^5}$. These results imply weaker congruences previously established by the author [Pacific J. Math. 2, 127–139 (1952); these Rev. 14, 138]. The proofs make use of the following two results which were not used in the earlier paper: first Glaisher's formula [Quart. J. Pure Appl. Math. 31, 1–35 (1899), §19] $A_{2i+1}^{(p)} \equiv \frac{1}{2}p(p-2i-1)A_{2i}^{(p)} \pmod{p^4}$ $(i > 1;\ p > 5)$, where

$$(x+1)(x+2)\cdots(x+m) = \sum_{r=0}^{m} A_r^{(m+1)} x^{m-r},$$

and second Nörlund's formula [Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924, p. 149]

$$(B+1)(B+2)\cdots(B+m) = m!/(m+1),$$

where it is understood that, after expansion of the left member, B^r is replaced by B_r. *A. L. Whiteman.*

Carlitz, L. A note on Bernoulli and Euler numbers of order $\pm p$. Proc. Amer. Math. Soc. 4, 178–183 (1953).

Put

$$(x/(e^x-1))^k = \sum_{m=0}^{\infty} B_m^{(k)} x^m/m!, \qquad B_m = B_m^{(1)},$$

$$(2/(e^x+1))^k = \sum_{m=0}^{\infty} C_m^{(k)} x^m/2^m m!, \qquad C_m = C_m^{(1)},$$

where the $B_m^{(k)}$ are the Bernoulli numbers of order k, and the $C_m^{(k)}$ are numbers closely related to the Euler numbers of order k. In the paper reviewed above the author has proved that $B_p^{(p)} \equiv -p^2(p-1)!/2 \pmod{p^5}$ for a prime $p > 3$. Now he obtains corresponding congruences for $B_p^{(-p)}$, $C_p^{(p)}$, $C_p^{(-p)}$. Three representative results are the following: (1) $\bar{A}_p \equiv p^3(p-1)! + 2p^3 B_{p-1}(p-1)^{-1} \pmod{p^5}$, where $\bar{A}_m$ is defined by

$$\bar{A}_m = (p!)^{-1} \sum_{r=0}^{p} (-1)^{p-r} \binom{p}{r} r^{m+p};$$

(2) $$C_p^{(p)} \equiv -p^2(p-1)C_{p-2} \pmod{p^4};$$

(3) $$C_p^{(-p)} \equiv p^2 C_{p-1}^{(-p)} + \tfrac{3}{2}p^4 C_{p-4} \pmod{p^6}.$$

The proofs are based upon standard summation formulas for the $B_m^{(k)}$ and $C_m^{(k)}$ [N. Nielsen, Traité élémentaire des nombres de Bernoulli, Gauthier-Villars, Paris, 1923; N. E. Nörlund, Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924]. *A. L. Whiteman* (Princeton, N. J.).

Lekkerkerker, C. G. Prime factors of the elements of certain sequences of integers. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 265–276, 277–280 (1953).

The author proves the following theorem: Let a and b be two nonvanishing integers satisfying $a^2+4b > 0$ and let ω and $\bar{\omega}$ be the roots of $x^2 - ax - b = 0$. Put $\mu_n = (\omega^n - \bar{\omega}^n)/(\omega - \bar{\omega})$ $(n = 0, 1, 2, \cdots)$. It is easy to see that the μ_n are integers. Then for each sufficiently large n there exists a prime g_n so that $g_n \mid \mu_n$ and $g_n \nmid \mu_m$ for $0 \leqq m < n$. The result of the author generalises a theorem of Bang [Dickson, History of the theory of numbers, vol. 1, Carnegie Inst. of Washington, 1919, p. 385]. *P. Erdős* (South Bend, Ind.).

Palamà, G. On a theorem of D. H. Lehmer concerning the Tarry-Escott problem. Scripta Math. **19**, 19–23 (1953).

The theorem in question concerns b sets of integers all having equal sums of like powers. The precise statement of the theorem is given in Lehmer, Scripta Math. **13**, 37–41 (1947); these Rev. **9**, 78. The original method of proof was by use of a certain generating function. Simpler proofs have been given by Wright [J. London Math. Soc. **23**, 279–285 (1948); these Rev. **10**, 510] and Mirsky [Scripta Math. **14**, 126–127 (1948); these Rev. **10**, 283]. The author gives a simple proof for the case $b=2$ and indicates how it may be extended to case of any b using a theorem of Tarry. A variant of Tarry's theorem is given. *D. H. Lehmer.*

***Šafarevič, I. R. A general duality law and its application in the theory of algebraic numbers.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 291–298. Akadémiai Kiadó, Budapest, 1952. (Russian and Hungarian)

An exposition of results of the author published in Doklady Akad. Nauk SSSR (N.S.) **64**, 25–28 (1949); Mat. Sbornik N.S. **26(68)**, 113–146 (1950) [these Rev. **11**, 230].

Flanders, Harley. The norm function of an algebraic field extension. Pacific J. Math. **3**, 103–113 (1953).

Let K/k be a field extension of degree n. Let $\omega_1, \cdots, \omega_n$ be a basis for K over k, and for $A \in K$ write

$$A=a_1\omega_1+\cdots+a_n\omega_n$$

with $a_i \in k$. A function $f: K\to k$ is called a polynomial function if there exists a polynomial $F(X_1, \cdots, X_n)$ with coefficients in k such that $f(A)=F(a_1, \cdots, a_n)$ for all $A \in K$. It is called norm-like if $f(AB)=f(A)f(B)$ and if $f(a)=a^n$ for $a \in k$. Flanders shows that the norm $N_{K/k}$ is the only norm-like polynomial function if k is infinite, and is the only one of degree at most n if k is finite. In order to do this he first proves that the factorization of the norm polynomial $N(X_1, \cdots, X_n)=N_{K/k}(\omega_1X_1+\cdots+\omega_nX_n)$ in the ring $k(X_1, \cdots, X_n)$ is of the form $N(X)=(P(X))^{n/m}$, where $m=\max\{[k(\theta):k]$ for $\theta \in K\}$ and $P(X)=P(X_1, \cdots, X_n)$ is irreducible over k. *J. Tate* (New York, N. Y.).

Chalk, J. H. H. A theorem of Minkowski on the product of two linear forms. Proc. Cambridge Philos. Soc. **49**, 413–420 (1953).

Consider the grid $\Lambda(C)$, i.e., the set of all points $(x_1, x_2)=(\alpha u+\beta v+c_1, \gamma u+\delta v+c_2)$ $(u, v=0, \pm1, \pm2, \cdots)$, which may be obtained from the lattice Λ of the points $(\alpha u+\beta v, \gamma u+\delta v)$ by a translation which brings O to the position $C=(c_1, c_2)$. Let $\Delta=|\alpha\delta-\beta\gamma|>0$. Then three points $X^{(1)}, X^{(2)}, X^{(3)}$ of $\Lambda(C)$ with coordinates $x_1^{(i)}, x_2^{(i)}$ $(i=1, 2, 3)$ and determinant

$$\pm\begin{vmatrix} x_1^{(1)} & x_2^{(1)} & 1 \\ x_1^{(2)} & x_2^{(2)} & 1 \\ x_1^{(3)} & x_2^{(3)} & 1 \end{vmatrix}=\Delta$$

form a generating set for the grid, i.e., each point $X \in \Lambda(C)$ can be written in the form $X=X_1+m(X_1-X_3)+n(X_2-X_3)$ (m, n integers). The author shows: For each grid $\Lambda(C)$ with $\Delta>0$ there is a set of generating points in the region $|x_1x_2| \leq \frac{1}{2}\Delta$. Unless both α/β and γ/δ are rational, there is an infinity of such sets even in the region $|x_1x_2|<\frac{1}{2}\Delta$ and the proof shows that each of these sets has a point in the region $|x_1x_2| \leq \frac{1}{4}\Delta$. Moreover, an infinity of (u, v) satisfy the inequalities $|x_1x_2|<\frac{1}{2}\Delta$, $|x_1|<\epsilon$, where ϵ denotes an arbitrary positive constant. These are extensions of well-known results by Minkowski [cf. the reviewer's Diophantische Approximationen, Springer, Berlin, 1936, Kap. VI and III20]. *J. F. Koksma* (Amsterdam).

Inkeri, K. Über einige Verschärfungen eines Minkowskischen Satzes. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **136**, 16 pp. (1952).

Let $f(x, y)$ be an indefinite quadratic form with real coefficients. Barnes [Quart. J. Math., Oxford Ser. (2) **1**, 199–210 (1950); these Rev. **13**, 16] proved that, if

$$\mu(f)=\max\{|f(1, 0)|, |f(0, 1)|, \min(|f(1, 1)|, |f(1, -1)|)\}$$

then $\frac{1}{4}\mu(f)$ is an admissible bound in Minkowski's theorem; i.e., for any real x_0, y_0 there exist integers x, y such that $|f(x+x_0, y+y_0)| \leq \frac{1}{4}\mu(f)$. The author gives another proof of this, and shows that there is a form $q(x, y)$, equivalent to $f(x, y)$, reduced (in the sense of Gauss), and with $\mu(q) \leq \mu(f)$. The author discusses the relations between this bound and those, which have been shown to be admissible in Minkowski's theorem, by Heinhold [Math. Z. **44**, 659–688 (1939)], Davenport [Nederl. Akad. Wetensch., Proc. **49**, 815–821 (1946); these Rev. **8**, 444], and the author [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **66** (1950); these Rev. **12**, 320]. *C. A. Rogers* (London).

Watson, G. L. Minkowski's conjectures on the critical lattices of the region $|x|^p+|y|^p \leq 1$. I. J. London Math. Soc. **28**, 305–309 (1953).

Minkowski's problem to find the critical lattices for the regions $|x|^p+|y|^p \leq 1$ $(p>1)$ is unsolved as yet. However, any critical lattice has six points on the boundary and no points inside (apart from the origin). Special lattices of this type are (i) the one which contains the point $x=1$, $y=0$, and (ii) the one containing $x=y=2^{-p}$. Their determinants are denoted by Δ_0, Δ_1, respectively. Probably the critical lattice is either of these, so it is desirable to know which of Δ_0, Δ_1 is the smaller. Minkowski conjectured that $\operatorname{sgn}(\Delta_0-\Delta_1)=\operatorname{sgn}(p-2)$ $(p>1)$; this was disproved by C. S. Davis [same J. **23**, 172–175 (1948); these Rev. **10**, 512, 856]. Davis also conjectured that there is a number p_0 (about 2.5725) such that $\operatorname{sgn}(\Delta_0-\Delta_1)=\operatorname{sgn}(p-2)(p_0-p)$ $(p>1)$. The present author proves this, with $2.57<p_0<2.58$. The discussion depends on the sign of

$$f(p)=2^{-p}(H+1)^p+2^{-p}(H-1)^p-2,$$

where $H^p=4(1-2^{-p})$. *N. G. de Bruijn* (Amsterdam).

Rankin, R. A. On positive definite quadratic forms. J. London Math. Soc. **28**, 309–314 (1953).

Let $f(x_1, \cdots, x_n)$ be a positive quadratic form in n variables and of determinant 1. Let $M_r(f)$ denote the lower bound of any principal minor of order r of any form equivalent to f $(1 \leq r \leq n-1)$. It is shown then that for given f the lower bound $M_r(f)$ is attained, that, if F is the form adjoint to f, we have $M_r(f)=M_{n-r}(F)$, and that $M_r(f)$ is bounded above for all forms of determinant 1 and attains its upper bound. If $\gamma_{n,r}$ denotes the maximum of $M_r(f)$ for all forms of determinant 1, then $\gamma_{n,m} \leq \gamma_{r,m}(\gamma_{n,r})^{m/r}$ for $1 \leq m<r \leq n-1$. The author further proves $\gamma_{4,2}=3/2$. From these results and the known values of $\gamma_{n,1}$ for $1 \leq n \leq 8$ he obtains upper bounds for $\gamma_{n,r}$ for small values of n and r.

J. F. Koksma (Amsterdam).

Coxeter, H. S. M., and Todd, J. A. An extreme duodenary form. Canadian J. Math. 5, 384–392 (1953).

A positive definite quadratic form of determinant Δ that assumes a minimum integral value M when its n arguments assume integral values not all zero is called extreme if Δ/M^n is a minimum for infinitesimal variations of the coefficients. The form which assumes the smallest minimum for a given n is called absolutely extreme. For $n=12$ a new extreme duodenary form K_{12} is described for which $\Delta(2/M)^{12}$ has a value 729/4096 that is less than the values 13, 4, 1, 1/4 for other known extreme duodenary forms, but it is not known whether K_{12} is absolutely extreme. Expressed as a sum of squares in variables $x_1, \cdots, x_6, y_1, \cdots, y_6$, the new form is

$$12K_{12}=\sum(6x_j+3y_j+2y_6)^2 +3(6x_6-y_6+2\sum y_j)^2+3\sum y_j^2+y_6^2,$$

where j is summed from 1 to 5. It is derived by considering a lattice of points whose coordinates are the Eisenstein integers $A_j\omega+B_j\omega^2$ in complex unitary six space. It is eutactic and perfect. *J. S. Frame.*

Erdös, Paul, and Rogers, C. A. The covering of n-dimensional space by spheres. J. London Math. Soc. 28, 287–293 (1953).

Given a system of equal spheres which cover n-dimensional space, one can define the density of the covering as the lower limit of the density of covering of a large cube as the cube expands. Let ϑ^*, depending only on n, denote the lower bound of the densities of all such coverings. The main result of the paper is that $\vartheta^*>16/15-\epsilon_n$, where $\epsilon_n\to 0$ as $n\to\infty$. A similar result, with 4/3 for 16/15, has been proved by Bambah and Davenport [same J. 27, 224–229 (1952); these Rev. 14, 787] for coverings in which the centres of the spheres are restricted to form a lattice. The present proof is again based on the consideration of the polyhedron Π consisting of all points which are nearer to the centre of one particular sphere than to the centre of any other sphere. The first difficulty is to estimate the number of faces of Π, and the authors show by an ingenious argument that this number is at most ϑ^*4^n-1. To complete the proof, an upper bound is needed for the proportion of the volume of an n-dimensional sphere which can be filled by a convex polyhedron with a given number of faces. A fairly simple argument on this question leads to the final result but with $(16/15)^{1/2}$ in place of 16/15. To obtain the latter, appeal is made to an inequality of Rogers concerning integrals over convex sets [same J. 28, 293–297 (1953); these Rev. 14, 965]. *H. Davenport* (London).

Prachar, Karl. Über höhere zahlengeometrische Minima. Arch. Math. 4, 39–42 (1953).

Let S be an n-dimensional set with the origin o as centre. For any positive integer l let $\Delta_l(S)$ be the lower bound of the determinants $d(\Lambda)$ of the lattices Λ having less than l point pairs $\pm\mathfrak{x}$ in S. The rth successive minimum $\sigma_{rl}=\sigma_{rl}(S,\Lambda)$ of order l of S for a lattice Λ is defined to be the lower bound of the positive numbers σ such that the set σS contains, not only at least r linearly independent points of Λ, but also at least l point pairs of Λ. If S is a convex body and has volume V, it was proved by Minkowski that

$$\sigma_{11}\sigma_{21}\cdots\sigma_{n1}V\leqq 2^n d(\Lambda)$$

and by Hlawka [J. Reine Angew. Math. 187, 246–252 (1950); these Rev. 12, 161] that $\sigma_{11}\sigma_{21}\cdots\sigma_{n1}V\leqq l2^n d(\Lambda)$. Chabauty [C. R. Acad. Sci. Paris 228, 796–797 (1949); these Rev. 10, 511] and Rogers [Nederl. Akad. Wetensch., Proc. 52, 256–263 (1949); these Rev. 10, 511; see also Mahler, ibid. 52, 633–642 (1949); these Rev. 10, 512] prove that in the general case $\sigma_{11}\sigma_{21}\cdots\sigma_{n1}\Delta_l(S)\leqq 2^{(n-1)/2}d(\Lambda)$. The author in this paper proves the result

$$\sigma_{11}\sigma_{21}\cdots\sigma_{n1}\Delta_l(S)\leqq 2^{(n-1)/2}d(\Lambda),$$

bearing the same relationship to Hlawka's result as that of Chabauty's and Rogers' bears to that of Minkowski. He further shows that $\sigma_{11}\sigma_{21}\cdots\sigma_{n1}\Delta_l(S)\leqq d(\Lambda)$ in the cases when (a) S is convex and two-dimensional, or (b) S is an n-dimensional sphere, or (c) S is an n-dimensional cube. *C. A. Rogers* (London).

Cohn, Harvey. Stable lattices. Canadian J. Math. 5, 261–270 (1953).

Let $\varphi(\mathbf{x})$ be a real continuously differentiable function, homogeneous of positive degree h, defined on the vectors of n-space. Let $\mathfrak{L}$ be a lattice with determinant Δ. Write $M(\mathfrak{L})=\inf_{\mathfrak{x}\in\mathfrak{L},\ \mathfrak{x}\neq(0)}|\varphi(\mathbf{x})|\Delta^{-h/n}$. Further, $M_0=\sup_{\mathfrak{L}} M(\mathfrak{L})$ ($M_0^{-n/h}$ is what is usually called the determinant of the region defined by $\varphi(\mathbf{x})\leqq 1$). A lattice $\mathfrak{L}_0$ is called critical if $M(\mathfrak{L}_0)=M_0$; in that case $\mathfrak{L}_0$ gives the absolute maximum of $M(\mathfrak{L})$. Relative maxima can also occur. The author defines a special type of relative maxima, which are furnished by what he calls stable lattices. The definition is somewhat complicated: Write $F=\varphi(\mathbf{x})\Delta^{-h/n}$; if the lattice point $\mathbf{x}^{(i)}$ of $\mathfrak{L}$ varies continuously with $\mathfrak{L}$, the variations of $\mathfrak{L}$ give rise to a differential $d|F^{(i)}|$. Let Q be the minimum number of lattice points $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(Q)}$ with the property that for every lattice point $\mathbf{x}^{(k)}$ the differential $d|F^{(k)}|$ is a linear combination of $d|F^{(1)}|, \cdots, d|F^{(Q)}|$. ($Q$ is called the free dimension, and is shown to be independent of $\mathfrak{L}$.) Now $\mathfrak{L}$ is called stable if $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(Q+1)}$ can be chosen in such a way that (i) $|F^{(i)}|=M(\mathfrak{L})>0$ ($i=1, \cdots, Q+1$), and (ii) in every non-trivial relation $A_1 d|F^{(1)}|+\cdots+A_{Q+1}d|F^{(Q+1)}|=0$ all A_i are positive.

The author investigates, with $\varphi=x_1\cdots x_n$, the stability of the lattices furnished by the numbers of a module in a totally real algebraic number field of degree n, relating the vector of its n conjugates to any number of the field (hence φ is the norm). Complex fields are also considered. *N. G. de Bruijn* (Amsterdam).

Peck, L. G. On uniform distribution of algebraic numbers. Proc. Amer. Math. Soc. 4, 440–443 (1953).

Let K be a real algebraic number field of degree $n+1$ over the rational field R, and let $1, \omega_1, \cdots, \omega_n$ be an R-base for K. If $f(x_1, \cdots, x_n)$ is Riemann integrable over the unit cube in n-space, and periodic with period 1 in all arguments, then Weyl's [Math. Ann. 77, 313–352 (1916)] metrical extension of the Kronecker approximation theorem asserts that

$$\sum_{m=0}^{M-1} f(m\omega_1, \cdots, m\omega_n)=\mathfrak{M}(f)+o(M),$$

where $\mathfrak{M}(f)$ is the average of f over the unit cube. The author proves that $o(M)$ can be replaced by $O(1)$ if f has the form

$$f(x_1, \cdots, x_n)=\sum_{q_1, \cdots, q_n=-\infty}^{\infty} a(q_1, \cdots, q_n) \times\exp\{2i\pi(q_1x_1+\cdots+q_nx_n)\},$$

where $a(q_1, \cdots, q_n)=O\{(|q_1|+\cdots+|q_n|)^{-n-c}\}$, $c>0$.

The proof depends on the convergence of the series $\sum_{\alpha\in\mathfrak{a},|\alpha|<C}|N(\alpha)|^{-1}(m(\alpha))^{-s}$, where $\mathfrak{a}$ is an integral ideal of K, and $m(\alpha)=\max(|\alpha^{(1)}|,\cdots,|\alpha^{(n)}|)$; here $\alpha^{(0)}$, $\alpha^{(1)}$, $\cdots$, $\alpha^{(n)}$ are the conjugates of $\alpha=\alpha^{(0)}$. In order to show the convergence the author applies the Dirichlet theory of units as well as the convergence of the Dedekind series $\zeta_K(s)$ $(s>1)$.

N. G. de Bruijn (Amsterdam).

Obrechkoff, N. Sur l'approximation diophantique linéaire pour des valeurs positives des variables. C. R. Acad. Bulgare Sci. 4 (1951), no. 1, 1–4 (1953). (Russian summary)

This is the same in substance as Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 46, 343–356 (1950); these Rev. 13, 921. *H. Davenport* (London).

***Korobov, N. M. On a question of diophantine inequalities.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 259–262. Akadémiai Kiadó, Budapest, 1952. (Russian. Hungarian summary)

After a brief historical account of some problems on uniform distribution the author proves the following result. If, for some function $\varphi(x)$ and arbitrarily chosen integers $m_1, m_2, \cdots, m_s$, not all zero,

$$m_1\varphi(x+1)+m_2\varphi(x+2)+\cdots+m_s\varphi(x+s)$$

is uniformly distributed, then αq^x is uniformly distributed, where q is an integer greater than unity and

$$\alpha=\sum_{k=1}^{\infty}[\{\varphi(k)\}q]/q^k.$$

This is actually a particular case of earlier results due to the author [Uspehi Matem. Nauk (N.S.) 4, no. 1(29), 189–190 (1949); Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 215–238 (1950); these Rev. 11, 231; 12, 321]. A general theorem giving sufficient conditions for a function $\varphi(x)$ to possess the above property is stated (see references quoted).

R. A. Rankin (Birmingham).

LeVeque, W. J. On n-dimensional uniform distribution modulo 1. Michigan Math. J. 1 (1952), 139–162 (1953).

Extending the method used in an earlier paper [Proc. Amer. Math. Soc. 1, 380–383 (1950); these Rev. 12, 163] the author generalizes results of Erdös and Koksma [Nederl. Akad. Wetensch., Proc. 52, 264–273, 851–854 = Indagationes Math. 11, 79–88, 299–302 (1949); these Rev. 11, 14, 331] on the discrepancy modulo 1 of sequences $f_1(x)$, $f_2(x)$, $\cdots$ $(a\leqq x\leqq b)$, by considering s-tuples of functions of r continuous variables $x_1, x_2, \cdots, x_r$ $(a_\rho\leqq x_\rho\leqq b_\rho,\ \rho=1,2,\cdots,r)$ and n sequential variables. The results are too complicated to be stated here. *J. F. Koksma* (Amsterdam).

ANALYSIS

Dresher, Melvin. Moment spaces and inequalities. Duke Math. J. 20, 261–271 (1953).

For given continuous functions $f_1, f_2, \cdots, f_n$ and for a variable distribution function $\Phi(x)$ the set $r=(r_1, r_2, \cdots, r_m)$, $r_i=\int_0^1 f_i(x)d\Phi(x)$ forms an mth moment space. For $f_1=f(x)^p$, $f_2=g(x)^{p/(p-1)}$, $f_3=f(x)g(x)$ $(0\leqq x\leqq 1,\ p>1)$, resp. $f_1=f(x)^p$, $f_2=g(x)^p$, $f_3=[f(x)+g(x)]^p$ $(0\leqq x\leqq 1,\ p>0)$, resp. $f_1=g(x)$, $f_2=f(x)g(x)$, $f_3=\psi[f(x)]g(x)$ $(0\leqq x\leqq 1,\ f(x)$ and $g(x)$ are continuous, $g(0)=0$, $\psi(x)$ is convex), resp. $f_1=f(x)^p$, $f_2=g(x)^p$, $f_3=[f(x)+g(x)]^p$, $f_4=f(x)^r$, $f_5=g(x)^r$,

$$f_6=[f(x)+g(x)]^r \quad (0\leqq x\leqq 1,\ p\geqq 1,\ 1>r>0),$$

the author obtains the integral inequalities of Hölder, Minkowski, Jensen, and the inequality

$$\left[\frac{\int_0^1(f+g)^p d\Phi}{\int_0^1(f+g)^r d\Phi}\right]^{1/(p-r)}\leqq\left[\frac{\int_0^1 f^p d\Phi}{\int_0^1 f^r d\Phi}\right]^{1/(p-r)}+\left[\frac{\int_0^1 g^p d\Phi}{\int_0^1 g^r d\Phi}\right]^{1/(p-r)}$$

respectively, making use of the following lemma: if $F_1(x)\geqq 0$, $F_2(x)>0$ are bounded and continuous $(0\leqq x\leqq 1)$, then, for given $0\leqq x_1<x_2<\cdots<x_n\leqq 1$ and $w_i>0$, $\sum_{i=1}^n w_i=1$, there exist a $w>0$ and an $x_1\leqq\bar{x}\leqq x_n$ such that

$$\sum_{i=1}^{n} w_i F_k(x_i)=wF_k(\bar{x}) \quad (k=1,2).$$

The author proves also that μ_a, μ_b, μ_c is a moment sequence, i.e., there exists a $\Phi(x)$ such that $\mu_a=\int_0^1 x^a d\Phi(x)$, etc., if and only if $0\leqq\mu_c\leqq 1$, $\mu_c^{b/c}\leqq\mu_b\leqq\mu_c$ and

$$(\mu_b^{a-c}/\mu_c^{a-b})^{1/b-c}\leqq\mu_a\leqq[(1-t^a)-(1-\mu_a)(1-t^c)]/(1-t^c),$$

where t is a solution of the equation

$$(1-t^b)(1-\mu_a)=(1-t^a)(1-\mu_b).$$

J. Aczél (Debrecen).

Danskin, J. M. Dresher's inequality. Amer. Math. Monthly 59, 687–688 (1952).

Using moment-space theory, M. Dresher, in the paper reviewed above, has proved the following result: If $p\geqq 1\geqq r\geqq 0$, $f(x)\geqq 0$, $g(x)\geqq 0$, and $\varphi(x)$ is a distribution function, then

$$\left[\frac{\int[f(x)+g(x)]^p d\varphi(x)}{\int[f(x)+g(x)]^r d\varphi(x)}\right]^{1/(p-r)}\leqq\left[\frac{\int[f(x)]^p d\varphi(x)}{\int[f(x)]^r d\varphi(x)}\right]^{1/(p-r)}+\left[\frac{\int[g(x)]^p d\varphi(x)}{\int[g(x)]^r d\varphi(x)}\right]^{1/(p-r)}.$$

This generalizes the known inequality [Beckenbach, same Monthly 57, 1–6 (1950); these Rev. 11, 422] obtained by setting $p=r+1$. The author now gives an alternative proof of Dresher's inequality, showing it to be a mélange of the Hölder and Minkowski inequalities. *E. F. Beckenbach.*

Selfridge, R. G. Approximations with least maximum error. Pacific J. Math. 3, 247–255 (1953).

Généralisation d'un théorème d'existence et d'unicité sur la meilleure approximation d'une fonction bornée dans un segment fini par des expressions généralisant les polynomes; exposé d'une méthode, dont la convergence n'est pas démontrée, pour le calcul de l'expression de la meilleure approximation; application numérique à la meilleure approximation de $\sin x$ dans $[0,1]$ par un polynome de la forme $ax+bx^3$. *J. Favard* (Grenoble).

Picone, Mauro. Allgemeine Gesichtspunkte zur Interpolation und einige durch sie angeregte Untersuchungen. Monatsh. Math. 57, 44–65 (1953).

Exposé très clair des méthodes introduites par l'auteur en calcul numérique [Ann. Scuola Norm. Super. Pisa (3) 5, 193–244 (1951); ces Rev. 14, 144] et qui permettent de rattacher à une même idée les diverses formules d'interpolation, ou de quadrature mécanique, et le calcul des erreurs,

tout en ouvrant la voie aux extensions. Exposé de nouveaux résultats sur l'approximation des fonctions holomorphes dans un domaine, et en particulier sur les développements en série de polynomes de Legendre. *J. Favard* (Grenoble).

Bajraktarević, M. Sur les bornes du module d'une somme. Bull. Soc. Math. Phys. Serbie **4**, no. 3–4, 17–27 (1952). (Serbo-Croatian. French summary)

From the French summary: Let $S_n(\theta)=\max T(\theta)\sin\frac{1}{2}k\theta$ for $0\leqq k\leqq n$, $0<|\theta|<\pi$, $0<a_0$, $T(\theta)=a_0/\sin\frac{1}{2}\theta$. Then always $S_n(\theta)\leqq T(\theta)$. If $|\theta|=(p/q)\pi$ where integers p, q $(0<p<q)$ have no common factors, then $S_n(\theta)=|T(\theta)|$ for $p=4r+j$ with $j=1$ and $q\leqq n$ or $j=3$ and $3q\leqq n$; also

$$S_n(\theta)=|T(\theta)|\sin\tfrac{1}{2}(q-1)\pi/q$$

for $p=2(2m+1)$ and $2q\leqq n$ or $p=4m$ and $q\leqq n$. For $|\theta|=A\pi$ with A irrational, $\lim S_n(\theta)=|T(\theta)|$ as $n\to\infty$, but with $A=1$, $S_n(\theta)=1$. Let $0\leqq a_n\leqq a_{n-1}\leqq\cdots\leqq a_0$ and $s_n=\sum a_k\exp(k\theta i)$, $k=0, 1, \cdots, n-1$; $n=0, 1, 2, \cdots$, $s_0=0$. Then, for n_0 a sufficiently large positive integer, $s_n\leqq S_n(\theta)$ when $0<\theta\leqq\pi$, $n_0\leqq n$. This inequality is not satisfied when n is taken very small. *M. Marden* (Milwaukee, Wis.).

Obrechkoff, Nikola. Sur quelques propriétés des fonctions réelles, définies sur le demi axe réel. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie II. **47**, 109–134 (1952). (Bulgarian. French summary)

The author develops Tauberian theorems analogous to those which are familiar for the behavior of a function and its derivatives at ∞, but which now apply instead to the functions defined recursively by

$$f_0(x)=f(x),\quad f_k(x)=f_{k-1}'(x)+\beta x f_{k-1}''(x),\quad \beta>1.$$

Then he proves a representation theorem for functions such that $f(\infty)$ exists and $f_n(x)\geqq 0$. This leads finally to the condition that all $f_n(x)\geqq 0$ as necessary and sufficient for the representation of $f(x)$ in the form $f(x)=\int_0^\infty\Phi(xt)d\omega(t)$, $\omega(t)$ bounded and nondecreasing, $\beta=1+\alpha$,

$$\Phi(x)=\int_0^\infty \exp\{xt-t^{-1}\}t^{-(\alpha/\beta)-1}dt.$$

[The representation theory for this kernel seems to be included in the general theory of Hirschman and Widder, Trans. Amer. Math. Soc. **67**, 69–97 (1949), pp. 87 ff.; these Rev. **11**, 350.] *R. P. Boas, Jr.* (Evanston, Ill.).

***Obrechkoff, Nikola. Sur quelques égalités limites pour les dérivées des fonctions et les différences des suites.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 595–612. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

The author generalizes theorems in which inferences are made from the behavior at ∞ of $f(x)$ and $f^{(n)}(x)$ to the behavior of $f^{(i)}(x)$ for $0<i<n$; the generalization consists in imposing the hypotheses only for a set of intervals and obtaining the conclusion for a set of somewhat smaller intervals. He also gives analogous results for sequences. Typical results: (a) Let $\phi(x)$ satisfy $\lim_{x\to\infty}\phi(\lambda x)/\phi(x)=\lambda^m$ for each positive λ. Let $x_k<x_k'<x_{k+1}$, $x_k'-x_k>\theta x_k$, $\theta>0$. In each (x_k, x_k') let $f(x)\sim A\phi(x)$, $f^{(n)}(x)>-Mx^{-n}\phi(x)$, where $\phi(x)$ is monotone (in the same sense) in all (x_k, x_k'). Then in the intervals $x_k(1+\epsilon)\leqq x\leqq x_k'(1-\epsilon)$,

$$f^{(i)}(x)\sim Am(m-1)\cdots(m-i+1)x^{-i}\phi(x),\quad 1\leqq i\leqq n-1.$$

(b) Let $\lambda_i<\lambda_i'\leqq\lambda_{i+1}$, $\lambda_i'/\lambda_i\geqq\theta>1$. Let the (C, k) sums of a series $\sum a_n$ converge to s as $n\to\infty$ in the intervals $\lambda_i<n\leqq\lambda_i'$ and let $a_n>-M/n$ for these n. Then the (C, j) sums converge to s for $1\leqq j\leqq k-1$ as $n\to\infty$ through the intervals $\lambda_i(1+\epsilon)\leqq n\leqq\lambda_i'(1-\epsilon)$. *R. P. Boas, Jr.*

Mandelbrojt, S. Quelques nouveaux théorèmes de fermeture. Ann. Soc. Polon. Math. **25** (1952), 241–251 (1953).

Let $f(x)$ be a function defined on $(-\infty, \infty)$ whose nth derivative satisfies $\int_{-\infty}^{\infty}|f^{(n)}(x)|dx<M_n$ $(n=0, 1, 2, \cdots)$. Let $g(x)$ be a function of L, $\{\nu_n\}$ a sequence of positive integers of lower density $>\frac{1}{2}$. The author gives conditions bearing on the sequences $\{M_n\}$ and $\{\nu_n\}$ and on the zeros of the Fourier transforms of $f(x)$ and $g(x)$ which insure that $f(x+a)$ is in the span of the $f^{(\nu_n)}(x)$ and the translates $g(x+b)$ of $g(x)$. The full statement is too long for reproduction. This work generalizes previous results by the author [J. Analyse Math. **1**, 180–208 (1951); these Rev. **13**, 540] which covered the case $\nu_n=n$. *W. H. J. Fuchs* (Ithaca, N. Y.).

***Klein, Felix. Elementary mathematics from an advanced standpoint. Arithmetic-algebra-analysis.** Translated by E. R. Hedrick and C. A. Noble. Dover Publications, Inc., New York, N. Y., 1953. ix+274 pp. Cloth $3.25; paper $1.50.

Photo-offset reprint; the original was published by Macmillan, New York, 1932.

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, Wacław. Sur l'extension d'un théorème de M. D. Pompeiu aux nombres transfinis. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **43** (1950), 1–3 (1952).

A natural number n is composite if and only if $n=a+b+c+d$, where a, b, c, d are natural numbers such that $ad=bc$. The author shows that if ν is an ordinal number such that $\nu=\alpha+\beta+\gamma+\delta$, where α, β, γ, δ are positive ordinals satisfying $\alpha\delta=\beta\gamma$, then ν is composite. The converse, however, does not hold (e.g., if $\nu=\omega+2$). In order that a transfinite ordinal ν be composite, it is necessary and sufficient that there exist positive ordinals α, β, γ, δ such that $\nu=\alpha+\beta+\gamma+\delta$ and either $\alpha\delta=\beta\gamma$ or each of the numbers α, β, γ, δ is greater than 1 and less than ν. *F. Bagemihl.*

Sierpiński, Wacław. Sur les produits infinis de nombres ordinaux. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **43** (1950), 20–24 (1952).

The author obtains results concerning rearrangements of infinite products $\prod_{i<\omega}a_i$ in the usual sense, where the factors are ordinal numbers, which are analogous to those which he has found [Fund Math. **36**, 248–253 (1949); these Rev. **12**, 14] for sums. In particular, for a given sequence $\{\alpha_i\}_{i<\omega}$ only a finite number of different values can be obtained by rearranging the product, and if the sequence is nondecreasing, then rearrangement of the product produces no change in its value. In the course of the proof it is shown that if α is a transfinite ordinal number such that $\xi\alpha=\alpha$ for $0<\xi<\alpha$, then, for any ordinal β, $\xi\beta\alpha=\beta\alpha$ for $0<\xi<\alpha$.

F. Bagemihl (Princeton, N. J.).

*Sierpiński, Wacław. **Sur les diviseurs de types ordinaux.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 397–399. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

The author, generalizing an earlier result of his [Fund. Math. **35**, 1–12 (1948), p. 8; these Rev. **10**, 358], shows that there exists a set F of $2^{\aleph_0}$ enumerable order types, such that each element of F is a right divisor of every other element of F. *F. Bagemihl* (Princeton, N. J.).

Sierpiński, Wacław. Sur les types ordinaux dont tous les vrais restes sont égaux. Fund. Math. **39** (1952), 1–7 (1953).

A proper remainder of an order type ϕ is an order type $\rho \neq 0$ for which at least one order type $\xi \neq 0$ exists such that $\phi = \xi + \rho$. (1) There exists at least one infinite order type all of whose proper remainders are equal to the order type ϕ if and only if $\phi = \omega^\alpha$, where α is a positive ordinal number. (2) In order that ψ be an order type all of whose proper remainders are equal to ω^α, where α is a given positive ordinal number, it is necessary and sufficient that either $\psi = \omega^\alpha$ or else $\psi = \cdots + \beta_3 + \beta_2 + \beta_1 + \omega^\alpha$, where the β_n are ordinal numbers such that $0 < \beta_n < \omega^\alpha$ for $n = 1, 2, \cdots$. (3) If n is a natural number, there are precisely $n+1$ order types all of whose proper remainders are equal to ω^α, and if $\omega_\gamma \leqq \alpha < \omega_{\gamma+1}$, there are precisely $\aleph_\gamma^{\aleph_0}$ order types all of whose proper remainders are equal to ω^α.

F. Bagemihl (Princeton, N. J.).

Denjoy, Arnaud. L'ordination des ensembles. C. R. Acad. Sci. Paris **236**, 1393–1396 (1953).

The author proves the following. (1) There are $2^{\aleph_0}$ continuous, increasing, real functions $f(x)$ of a real variable, such that $f(x)$ is rational or irrational according as x is rational or irrational. (2) If each of the mutually exclusive sets E_n $(n < \omega)$ is an enumerable, everywhere dense subset of the open unit interval, then there are $2^{\aleph_0}$ continuous, increasing functions $f(x)$ defined on the unit interval, such that $f(E_n) = E_n$ $(n < \omega)$. (3) If with every ordinal number ξ of the second number class there is associated an ordinal $\varphi(\xi) < \xi$, then there is a value of φ which is assumed for $\aleph_1$ ordinals of the second number class. These results or generalizations of them are known [see, e.g., Skolem, Skr. Vid.-selsk. Kristiania. I. Mat.-Nat. Kl. 1920, no. 4; Franklin, Trans. Amer. Math. Soc. **27**, 91–100 (1925); Neumer, Mat. Z. **54**, 254–261 (1951); these Rev. **13**, 331].

F. Bagemihl (Princeton, N. J.).

Ginsburg, S. On the distinct sums of λ-type transfinite series obtained by permuting the elements of a fixed λ-type series. Fund. Math. **39** (1952), 131–132 (1953).

Denote by $N(\sum_{\xi<\omega_\rho}\alpha_\xi)$, where the α_ξ are ordinal numbers, the number of distinct values of all rearrangements $\sum_{\xi<\omega_\rho}\alpha_\xi'$ of the first series. Generalizing results of Sierpiński [Fund. Math. **36**, 248–253 (1949); these Rev. **12**, 14], the author shows: (1) if ω_ρ is regular and $\rho > 0$, then $N(\sum_{\xi<\omega_\rho}\alpha_\xi) \leqq \sum_{\xi<\omega_\rho}\aleph_\rho^{\aleph_\xi}$; (2) there exist α_ξ such that $N(\sum_{\xi<\omega_{\alpha+1}}\alpha_\xi) = \aleph_{\alpha+1}$; (3) if ω_ρ is regular, and $\{\alpha_\xi\}_{\xi<\omega_\rho}$ is non-decreasing, then $N(\sum_{\xi<\omega_\rho}\alpha_\xi) = 1$. *F. Bagemihl.*

Ginsburg, Seymour. Real-valued functions on partially ordered sets. Proc. Amer. Math. Soc. **4**, 356–359 (1953).

Let P be a partially ordered system with no minimal elements. For each $p \in P$ let $g(p) = \{p' \mid p' \leqq p\}$ and let $Q = \{q \mid q$ a coinitial subset of a $g(p)\}$: order Q by $\subseteq$. The reviewer showed [Duke Math. J. **11**, 201–229 (1944); these Rev. **5**, 231] that for each such P the directed system of coinitial residual subsets of P is as effective for Alaoglu-Birkhoff convergence as is P, and showed that for each cardinal number α there is an everywhere branching system E as effective for A-B convergence as every P of cardinal $\leqq \alpha$.

Let h be a function from Q to P such that $h(g(p)) = p$ for each p and $h(q) \in q$ for each q. This note observes that Q is everywhere branching and that g is an isomorphism of P into Q. Let X be a neighborhood space, x a point of X, and f a function from P into X. Theorem 1 is the special case, X = real numbers, of: $\lim_{p \in P} f(p) = x$ if and only if $\lim_{q \in Q'} f(h(q)) = x$. Theorem 2 follows from the case X = real numbers, P directed, of: If $P' \in Q$ and if $Q' = \{q' \mid q' \in Q$ and $q' \subseteq P\}'$, then $\lim_{p \in P'} f(p) = x$ if and only if $\lim_{q \in Q'} f(h(q)) = x$.

M. M. Day (Urbana, Ill.).

Grossmann, Aleksandar. Sur une propriété des ensembles ordonnés. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 24–26 (1953). (Serbo-Croatian summary)

Let ξ be an arbitrary ordinal number, and E be an ordered set with $|E| \leqq \aleph_\xi$. It is shown that either $\Gamma(E)$ (i.e., the upper bound of the ordinals of all well-ordered subsets of E) is less than $\omega_{\xi+1}$, or else E is an ordered sum of sections of E, over an ordered set J such that, for every interval H of J, $\Gamma(H) = \omega_{\xi+1}$. This was proved by Shepherdson [Proc. London Math. Soc. (3) **1**, 291–307 (1951), p. 304, Corollary; these Rev. **13**, 330]. *F. Bagemihl* (Princeton, N. J.).

Lyapunov, A. A. Separability and nonseparability of R-sets. Mat. Sbornik N.S. **32(74)**, 515–532 (1953). (Russian)

As sequel to some of his previous papers and in connection with researches of Lusin, Novikov, etc., the author studies the questions of separation of various kinds of sets by means of sets of another kind. As in the particular case of A-sets, one has 2 types of separation: the simple one (Lusin) and the multiple one (Novikov); in each of these cases one has to distinguish 2 types of separation, according as the intersection of the given sets to be separated is void or nonvoid. For example, for the simple separation one has the Lusin principles: (1) disjoint A-sets are B-separable (type 1); (2) if X, Y are A-sets, then $X - X \cap Y$, $Y - X \cap Y$ are separable by B-sets (type 2). The corresponding propositions hold true for what the author calls the ordinary $R\mathfrak{A}$-operations (Theorems I, II). [For the terminology and notations one is referred especially to an unavailable paper of the author: Trudy Mat. Inst. Steklov. **40** (1953).] As a corollary to Theorem II one has the fact that if $s \in \{A, R_\sigma, R_{\sigma\delta}\}$, then any pair of disjoint Z-sets can be separated by means of BZ-sets [cf. Lusin, Leçons sur les ensembles analytiques, Gauthier-Villars, Paris, 1930, p. 210 for $Z = A$]. Let λ be a regular class of transfinite functions and ϕ_N a δ_s-function such that $\Xi(\lambda)$ is invariant with respect to all δ_s-functions ϕ_{N^*}; then for each sequence $\{E_n\}$ of $\Xi(\lambda)$-sets there exists a sequence $\{H_n\}$ of $C\Xi(\lambda)$-sets satisfying

$$H_n \supseteq E_n - \phi_{N^*}(\{E_n\}, \phi_N(\{H_n\})) = 0$$

(Theorem III). The proof is based upon the following Lemma: Let $\{\beta_n(x)\}$ be a sequence of transfinite functions and ϕ_N a δ_s-function; if

$$E_n = [\beta_n(x) = \omega_1],\ Q_{nm} = [\beta_n(x) \leqq \beta_m(x)],\ H_n = C\phi_{N^*}(\{Q_{nm}\}),$$

then $H_n \supseteq E_n - \phi_{N^n}(\{E_m\})$, $\phi_N(\{H_n\}) = 0$. The Novikov results concerning the existence of disjoint non-B-separable CA-sets are reasonably extended to R-sets too, for both types of separation (cf. Theorems V, VI, VII). Definition: Let $\lambda = \{\beta(x)\}$ be a class of transfinite functions (i.e., a mapping of a space into the class of ordinals); then the class of sets of the form $[\beta(x) = \omega_1]$ with $\beta \in \lambda$ is designated by $\mathfrak{Z}(\lambda)$; λ is called regular if

$$\beta_1, \beta_2 \in \lambda \text{ imply } [\beta_1(x) \geqq \beta_2(x)] \in \mathfrak{Z}(\lambda).$$

For a basis N of a δ_σ-operation, N^n means the set of all the $\eta \in N$ such that $n \in \eta$. For some basic definitions the author refers to his as yet unavailable article mentioned above; this makes the reading of the present paper difficult.

G. Kurepa (Zagreb).

Volkmann, Bodo. Über Hausdorffsche Dimensionen von Mengen, die durch Zifferneigenschaften charakterisiert sind. I. Math. Z. **58**, 284–287 (1953).

If g, e_n are integers and $0 \leqq e_n < g$, the set of fractions $\sigma = \sum f_n g^{-n}$ for which $f_n \leqq e_n$ (all n) is considered.

H. D. Ursell (Leeds).

***Surányi, János. Sur la structure des classes finies d'ensembles.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 401–407. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and French summaries)

If F is a subset of a set E, write $F^{+1} = F$ and $F^{-1} = E - F$. If E_j and F_j are subsets of E, $j = 1, \cdots, n$, the class $\{E_1, \cdots, E_n\}$ is defined to be equivalent to the class $\{F_1, \cdots, F_n\}$ if and only if there is a permutation π of the first n positive integers such that $\bigcap_{j=1}^{n} E_j^{\epsilon_j}$ and $\bigcap_{j=1}^{n} F_{\pi(j)}^{\epsilon_j}$ are empty and non-empty at the same time, for all sequences $\{\epsilon_1, \cdots, \epsilon_n\}$ with $\epsilon_j = \pm 1$, $j = 1, \cdots, n$. Let $N(n)$ be the number of equivalence classes of classes consisting of not more than n sets. The author proves that $n!N(n)/2^{2^n} \to 1$ as $n \to \infty$.

P. R. Halmos (Chicago, Ill.).

***Rudin, Walter. Principles of mathematical analysis.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. ix+227 pp. $5.00.

The author deals with the classical elements of real variable theory: Dedekind real numbers, elementary set-theory, convergence, continuity, differentiation, the Riemann-Stieltjes integral, uniform convergence, functions of several variables, Lebesgue measure and integrals. The presentation is intended for the undergraduate (or immediately post-graduate) reader, and is clear and concise: the necessity for restrictive hypotheses in theorems is emphasised by illustrative counter-examples. In general, the spaces considered are Euclidean, but the reader is introduced to more abstract ideas: for example, general metric spaces and measure spaces. [The reviewer has noticed only one lapse from a high standard of accuracy; this occurs in the chapter on the Riemann-Stieltjes integral. Two definitions, (6.2) and (6.26), of the integral are given and it is stated that they are equivalent if the integrator is monotonic. This is untrue unless the integrator is also supposed to be continuous. The initial error is in Theorem 6.11, where the same additional hypothesis must be made.]

U. S. Haslam-Jones (Oxford).

***Gončarov, V. L. Elementary functions of a real variable. Limits of sequences and functions. The general concept of a function.** Enciklopediya èlementarnoĭ matematiki. Kniga III. Funkcii i predely (osnovy analiza). [Encyclopaedia of elementary mathematics. Book III. Functions and limits (the foundations of analysis).] Pp. 9–296. Gosudarstv. Izdat. Tehn-Teoret. Lit., Moscow-Leningrad, 1952. 559 pp. 13.10 rubles.

This is in part a survey of elementary functions and their graphs such as is customarily found in textbooks on analytic geometry, and in part a conventional introduction to the theory of functions of a real variable.

R. P. Boas, Jr. (Evanston, Ill.).

Haupt, Otto, und Pauc, Christian Y. Halobedingungen und Vitalische Eigenschaft von Somensystemen. Arch. Math. **4**, 107–114 (1953).

In a previous article [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1950, 187–207 (1951); these Rev. **14**, 544] the authors stated sufficient conditions for the strong and weak Vitali properties of systems of somas belonging to a Boolean σ-algebra, but postponed the proofs. Now these proofs are given in the present paper.

A. Rosenthal.

Alexiewicz, A. On some theorems of S. Saks. Studia Math. **13**, 18–29 (1953).

Let X and Y be F spaces, T an abstract set and $\mathfrak{E}$ a σ-ring of subsets of T on which is defined a countably additive measure μ with $\mu(T) < \infty$. Assume that the measure space (T, E, μ) is separable in the topology defined by the metric $d(e_1, e_2) = \mu(e_1 + e_2 - e_1 e_2)$. Let $U(x, t)$ denote an operation from $X \times T$ to Y satisfying

(a) $$U(x_1 + x_2, t) = U(x_1, t) + U(x_2, t)$$

a.e. in T. (b) $x_n \to 0$ implies $\|U(x_n, t)\| \to 0$ in measure on T. (c) If $S \subset T$ is open, for any $x \in X$, $E_t\{U(x, t) \in S\}$ is measurable. Under these conditions it is shown that if R is a Borel measurable linear manifold in Y there exists a decomposition $T = A + B$; $AB = 0$; $A, B \in \mathfrak{E}$ with (i) $U(x, t) \in R$ for all $x \in X$ and almost all $t \in A$, (ii) $U(x, t)$ non-$\in R$ for almost all $t \in B$ and all $x \in X$ except for a fixed set of first category. By applying this theorem for arbitrary X and for Y the space of measurable functions, the author deduces various theorems as special cases, among them certain theorems of Saks [Fund. Math. **10**, 186–196 (1927); Trans. Amer. Math. Soc. **35**, 549–556 (1933); **41**, 160–170 (1937)].

R. E. Fullerton (Madison, Wis.).

Marczewski, E., et Ryll-Nardzewski, C. Sur la mesurabilité des fonctions de plusieurs variables. Ann. Soc. Polon. Math. **25** (1952), 145–154 (1953).

If, for each t_0, $f(x, t_0)$ is α and, for each x_0, $f(x_0, t)$ is β, is $f(x, t)$ necessarily measurable L? $\alpha = \beta =$ upper semi-continuous is not enough (Sierpiński) but $\alpha =$ measurable L suffices if $\beta =$ continuous (Lebesgue) or monotone (Ursell). The authors give a theorem embracing both these results: their β is satisfied also by any $\phi(t)$ continuous at each point to one side or the other. They give also analogous work for Borel measurability and the Baire property.

H. D. Ursell (Leeds).

***Marczewski, Edward. Théorème ergodique; généralisations et applications.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 125–130. Akadémiai Kiadó, Budapest, 1952. (French and Hungarian. Russian summary)

An exposition of papers by Hartman, Marczewski, and Ryll-Nardzewski [Colloquium Math. **2**, 109–123 (1951);

these Rev. 13, 758], Ryll-Nardzewski [Studia Math. 12, 65–73, 74–79 (1951); these Rev. 13, 757] and Hartman [ibid. 12, 271–278 (1951); these Rev. 13, 758].

Calderon, A. P. A general ergodic theorem. Ann. of Math. (2) 58, 182–191 (1953).

Let $\mathfrak{G}$ be a locally compact group of one-to-one measure-preserving transformations in a finite measure space $\mathfrak{E}$. For each $t>0$ let N_t be a compact symmetric neighborhood of the identity in $\mathfrak{G}$ with $N_tN_s\subset N_{t+s}$. The author considers, under suitable measurability restrictions on $\mathfrak{G}$, averages of the form $p_t(F,x)=|N_t|^{-1}\int_{N_t}F(gx)dg$ where dg and $|N_t|$ represent Haar measure, and F an integrable function in $\mathfrak{E}$. It is assumed that $|N_{2t}|<\alpha|N_t|$ where α is a constant independent of t. It is shown that there is a subset S of the reals, of density 1, such that $p_t(F,x)$ converges for almost all x in $\mathfrak{E}$ as t approaches infinity through S. The dominated ergodic theorem is proved for functions $F\in L_p(\mathfrak{E})$, $p>1$. A similar setting is given for a general form of the mean ergodic theorem. These results contain the previously proved results in the case of continuous or discrete abelian groups with a finite number of generators. In the non-commutative case a result similar to one proved by the reviewer [Acta Sci. Math. Szeged 14, 1–4 (1951); these Rev. 13, 49] and Zygmund [ibid. 14, 103–110 (1951); these Rev. 13, 661] independently is given. *N. Dunford.*

Riesz, M. Court exposé des propriétés principales de la mesure de Lebesgue. Ann. Soc. Polon. Math. 25 (1952), 298–308 (1953).

Expository paper. *P. R. Halmos* (Chicago, Ill.).

***Riesz, Frigyes. Les ensembles de mesure nulle et leur rôle dans l'analyse.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 205–224. Akadémiai Kiadó, Budapest, 1952. (Hungarian and French. Russian summary)

Historical and expository lecture. *P. R. Halmos.*

Rényi, Alfréd. On a conjecture of H. Steinhaus. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 37–44 (1953). (Hungarian)

Hungarian version of the author's paper reviewed below. *P. R. Halmos* (Chicago, Ill.).

Rényi, Alfred. On a conjecture of H. Steinhaus. Ann. Soc. Polon. Math. 25 (1952), 279–287 (1953).

A countable family $\{f_n\}$ of bounded measurable functions on the unit interval is called maximal whenever there exists a measurable set Z of measure zero such that if x and y do not belong to Z and if $f_n(x)=f_n(y)$ for all n, then $x=y$. The principal theorem of the paper asserts that if $\{f_n\}$ is maximal, then (*) the set of all finite products of positive integral powers of the f_n's is complete in L^2. The family $\{f_n\}$ is called saturated (with respect to stochastic independence) if it is stochastically independent but is not properly contained in any other stochastically independent set. The conjecture of Steinhaus (referred to in the title) was that if $\{f_n\}$ is saturated, then (*) holds. This is disproved by a family of one term: $f_1(x)=x$ or 1 according as $0\leq x\leq\frac{1}{2}$ or $\frac{1}{2}<x\leq 1$. The connection between Steinhaus' insufficient condition and the author's sufficient one is as follows: if $\{f_n\}$ is independent and maximal, then it is saturated. *P. R. Halmos* (Chicago, Ill.).

***Rényi, Alfréd. Stochastical independence and complete systems of functions.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 299–316. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian. English summary)

The result of this paper is a special case of the result of the paper reviewed above. *P. R. Halmos* (Chicago, Ill.).

Maharam, Dorothy. The representation of abstract integrals. Trans. Amer. Math. Soc. 75, 154–184 (1953).

By an "abstract integral" is meant a linear, countably additive, order-preserving mapping φ from one function space F to another F'. Here F and F' are considered modulo given classes of null functions and are determined by corresponding Boolean algebras E and E'. They are assumed to satisfy a countable chain condition. Ordinary numerical integrals are included as the special case in which E' consists of a single atom. A general type of "F'-integral", i.e., having range in F', is exemplified by taking $E=(J,m)\times E'$, where (J,m) is a non-atomic σ-finite numerical measure algebra, and defining $\varphi(f)=\int f(x,y)dm(x)$ by partial integration on a product space representing E. The main theorem asserts that any F'-integral is isomorphic (in an appropriate sense) to a mapping φ of the type just described, where in general E must be restricted to a subalgebra of a principal ideal in the direct product $(J,m)\times E'$. In a sense, this gives a complete description of F'-integrals. A condition which characterizes those F'-integrals for which E need not be so restricted is found. Many auxiliary results connecting abstract integrals with abstract measure algebras, or concerning the embedding of one in another, or clarifying the relations between different possible meanings of "isomorphism" are also obtained. The methods are largely based on the author's theory of abstract-valued measures [same Trans. 65, 279–330 (1949); these Rev. 10, 519]. *J. C. Oxtoby.*

Lévy, P. Intégrales de Stieltjes généralisées. Ann. Soc. Polon. Math. 25 (1952), 17–26 (1953).

The author gives a general discussion and summary of his work in defining the integral $\int g(t)df(t)$ by probability methods, using Riemann-Stieltjes sums with random points of the integration interval. [Cf. Ann. Univ. Lyon. Sect. A. (3) 4, 67–74 (1941); C. R. Acad. Sci. Paris 212, 1066–1068 (1941); 229, 644–646 (1949); these Rev. 8, 37; 5, 126; 11, 166.] *J. L. Doob* (Urbana, Ill.).

***Mambriani, Antonio. Su la derivazione d'ordine qualsiasi.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 142–150. Casa Editrice Perrella, Roma, 1953.

The author defines a definite derivative of arbitrary order ν to be the ordinary Riemann-Liouville derivative, and an indefinite derivative of order ν to be a definite derivative (with some fixed origin x_0) plus an arbitrary "νth derivative of 0"; by this last he understands 0, when ν is a nonnegative integer; a polynomial of degree $-\nu-1$, when ν is a negative integer; and a function $(x-x_0)^{-\nu}\phi(x)$ with $\phi(x)$ regular and vanishing at ∞, for other values of ν. No applications are given. *R. P. Boas, Jr.* (Evanston, Ill.).

Temple, G. La théorie de la convergence généralisée et des fonctions généralisées et leurs applications à la physique mathématique. Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 11, 111–122 (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 356 (1953).

Exposé rapide de la théorie des distributions et de quelques applications physiques. *L. Schwartz* (Paris).

König, Heinz. Neue Begründung der Theorie der "Distributionen" von L. Schwartz. Math. Nachr. **9**, 129–148 (1953).

L'auteur définit une nouvelle théorie des "distributions" sur R^n [L. Schwartz, Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; ces Rev. **12**, 31, 833]. L'originalité réside dans l'extension de la notion de distribution, mais surtout dans la méthode axiomatique d'introduction, comparable à la méthode d'extension algébrique d'un corps ou à la méthode de Bochner [Vorlesungen über Fouriersche Integrale, Akademische Verlagsgesellschaft, Leipzig, 1932, pp. 110–144]. Nous supposerons $n=1$, ce qui simplifie notablement. V sera l'espace vectoriel des séries formelles $\sum_{k=0}^{\infty} f_k z^k$, suivant les puissances de z, ne contenant localement qu'un nombre fini de termes non nuls; chaque f_k est une classe (au sens de Lebesgue) de fonctions numériques f_k (non nécessairement localement sommable). Une dérivation D est définie dans V par $D(\sum f_k z^k) = \sum f_k z^{k+1}$. U sera le sous-espace vectoriel de V constitué des éléments qui localement sont combinaisons linéaires finies d'éléments de la forme $(g - Gz)z^k$, où g est localement sommable et G intégrale indéfinie de g; U est stable par D, de sorte que D opère sur le quotient $F = V/U$. L'application canonique de V dans F, restreinte au sous-espace des éléments $f = fz^0$, est biunivoque, de sorte que les éléments de F constituent une généralisation de la notion de classe de fonctions avec une dérivation toujours possible généralisant la dérivation usuelle. Les "distributions" peuvent être identifiées aux éléments d'un sous-espace L de F, ceux qui sont images dans F des éléments de V de la forme $\sum f_k z^k$, où les f_k sont localement sommables. L'auteur définit ensuite une notion de convergence (ne correspondant pas à une topologie) dans V, pour laquelle U est fermé, donc une notion de convergence dans F; et une multiplication des éléments de V par les fonctions indéfiniment différentiables, pour laquelle U est stable, donc une multiplication des éléments de F; ces diverses opérations sont reliées par les règles habituelles. Enfin on introduit une notion de localisation et de support. Par contre la convolution, les transformations de Fourier, Laplace, etc., ne peuvent se définir que sur le sous-espace L. Dans F la division par une fonction indéfiniment différentiable presque partout différente de 0 est un problème trivial (notre "problème de la division" revient alors à se demander quand le quotient d'un élément de L est encore dans L).

L. Schwartz (Paris).

Fürst, Dario. Eine Verallgemeinerung eines Satzes von Weyl. Arch. Math. **4**, 115–120 (1953).

The theorem of Weyl referred to in the title asserts that a necessary and sufficient condition for convergence in measure is the validity of the Cauchy condition in measure. The generalization involves, instead of sequences of functions, sequences of sets of functions; the inequalities defining convergence in measure are required to hold for an arbitrary function in the appropriate set of the sequence. Some emphasis is given to the fact that the axiom of choice is not used in the proof. *P. R. Halmos* (Chicago, Ill.).

Scorza Dragoni, Giuseppe. Un criterio di convergenza in lunghezza e la derivazione per serie. Rend. Sem. Mat. Univ. Padova **22**, 177–180 (1953).

A considerably shortened and simplified proof of a theorem of E. Baiada [Ann. Scuola Norm. Super. Pisa (3) **6**, 59–68 (1952); these Rev. **14**, 628]. *T. A. Botts.*

Medvedev, Yu. T. Two criteria of compactness of families of functions. Doklady Akad. Nauk SSSR (N.S.) **90**, 337–340 (1953). (Russian)

Theorem 1. Suppose X and Y are non-empty sets, M is a totally bounded metric space, and ϕ maps $X \times Y$ into M. Let $F = \{f_y : y \in Y\}$ and $G = \{g_x : x \in X\}$, where $f_y = \phi(x, y) \mid x \in X$ and $g_x = \phi(x, y) \mid y \in Y$. Then (with the usual metrization) F is totally bounded if and only if G is totally bounded. By suitable specialization this yields the classical theorem of Arzelà.

Theorem 2. Suppose X is a measurable subset of Euclidean space, with $mX < \infty$, and F is a set of measurable real functions on X. Then F is (relatively) compact with respect to convergence in measure if and only if for each $\epsilon > 0$ there is a transformation $f_\epsilon \mid f \in F$ of F into the space CX of bounded continuous real functions on X such that $\{f_\epsilon : f \in F\}$ is relatively compact and $m\{x : fx \neq f_\epsilon x\} < \epsilon$ for each $f \in F$.

V. L. Klee (Seattle, Wash.).

Leĭbenzon, Z. L. Investigation of certain properties of a continuous point transformation of an interval onto itself which have application in the theory of nonlinear oscillations. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 351–360 (1953). (Russian)

The author deals with rather superficial properties of a mapping $x^* = f(x)$ of the unit-segment into itself. He generally assumes (without stating it explicitly) that $f'(x)$ exists. He also considers pairs of points x, x^* such that $x^* = f(x)$, $f(x) = x^*$. No relationships with oscillations are discussed in the paper.

S. Lefschetz (Princeton, N. J.).

Cecconi, Jaurès. Sui teorema di Stokes. Rivista Mat. Univ. Parma **3**, 233–264 (1952).

Let S be an oriented Fréchet surface given in terms of a representation $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $(u, v) \in Q$, where Q is the unit square $0 \leq u \leq 1$, $0 \leq v \leq 1$, and let C be the boundary curve of S, given by the preceding equations restricted to the perimeter of Q. Finally, let $F(x, y, z)$ be a function defined in an open set O containing S. Assume that the following conditions are satisfied. (i) The Lebesgue area of S is finite. (ii) The length of C is finite. (iii) F and its first partial derivatives F_y and F_z are continuous in O. Then $\int\int_S F_x dz dx - \int\int_S F_z dx dy = \int_C F dx$, where these integrals are Weierstrass-type integrals (as defined by Tonelli and Cesari). The proof utilizes many deep results in the theory of surface area.

T. Radó (Columbus, Ohio).

Hosemann, R., und Bagchi, S. N. Begründung einer Algebra physikalisch beobachtbarer Funktionen mittels Faltungsoperationen. I. Präzisionen, Funktionenkomplex, Punktfunktionen, Fourier-Transformation eines Komplexes, Integro-Differentialoperator. Z. Physik **135**, 50–84 (1953).

Functions of a real variable are considered as defined within a non-vanishing "experimental error", the independent variable being restricted to a finite "domain of observation". For such functions, and in a non-rigorous way, the paper discusses various topics: Fourier transform, "point functions" defined as the analogues of Dirac's δ-functions, differentiation and integration operators. *L. Van Hove.*

Theory of Functions of Complex Variables

***Saks, Stanisław, and Zygmund, Antoni. Analytic functions.** Translated by E. J. Scott. Monografie Matematyczne, Tom XXVIII. Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1952. viii+451 pp. $6.50.

This is a translation of the authors' Funkcje Analityczne [vol. X of the same series, 1938], a text which, in spite of linguistic difficulties, obtained considerable fame outside Poland for its thoroughness, clarity, and originality of presentation. The authors have drawn on both the "geometric" and "arithmetic" approaches to the subject, and cover considerably more material than is usual in an introductory text. One of the novel features is the early introduction of Runge's theorem and its application to facilitate the proof of Cauchy's theorem and other results. Conformal mapping is discussed relatively late. The general part of the book is followed by three chapters, one on entire and meromorphic functions, one on elliptic functions, and one on $\Gamma(s)$, $\zeta(s)$ and Dirichlet series. The translation seems unnecessarily unidiomatic. *R. P. Boas, Jr.*

***Leja, Franciszek. Funkcje analityczne i harmoniczne. Tom I. [Analytic and harmonic functions. Vol. I.]** Monografie Matematyczne, Tom XXIX. Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1952. iv+174 pp. Zł. 15.00.

Traité élémentaire de la théorie des fonctions analytiques et harmoniques. Correspond, à peu près, à la partie du traité de Goursat traitant de la partie élémentaire de la théorie des fonctions analytiques. Exposé concis, rigoreux et assez complet. *S. Mandelbrojt* (Houston, Tex.).

***Gončarov, V. L. Elementary functions of a complex variable.** Ènciklopediya èlementarnoĭ matematiki. Kniga III. Funkcii i predely (osnovy analiza). [Encyclopaedia of elementary mathematics. Book III. Functions and limits (the foundations of analysis).] Pp. 491–552. Gosudarstv. Izdat. Tehn-Teoret. Lit., Moscow-Leningrad, 1952. 559 pp. 13.10 rubles.

A brief introduction to complex variables, containing only enough general theory to show the behavior of the elementary functions in the complex domain.

R. P. Boas, Jr. (Evanston, Ill.).

***Knopp, Konrad. Problem book in the theory of functions. Vol. II. Problems in the advanced theory of functions.** Translated by F. Bagemihl. Dover Publications, Inc., New York, N. Y., 1953. 138 pp. Cloth $2.50; paper $1.25.

Translation of Aufgabensammlung zur Funktionentheorie, Teil II, Aufgaben zur höheren Funktionentheorie [4th ed., de Gruyter, Berlin-Leipzig, 1949].

Zin, Giovanni. Esistenza e rappresentazione di funzioni analitiche, le quali, su una curva di Jordan, si riducono a una funzione assegnata. Ann. Mat. Pura Appl. (4) **34**, 365–405 (1953).

Let D be a region bounded by a rectifiable Jordan curve C, and let $\varphi(z)$ be Lebesgue integrable on C and satisfy the conditions $\int_C \varphi(z)z^n dz=0$ $(n=0, 1, 2, \cdots)$. The principal result is then that the function $f(z)$ defined by the conditions $f(z)=\varphi(z)$ on C, $f(z)=(2\pi i)^{-1}\int_C \varphi(t)(t-z)^{-1}dt$ in D, is continuous at every point P of C where (i) $\varphi(z)$ is continuous, (ii) P is Lipschitzian (i.e., P belongs to an arc AB of C such that, at each point of AB, the function representing AB satisfies a Lipschitz condition). The author then considers the problem of finding an analytic function in D which reduces to $\varphi(z)$ when $\varphi(z)$ is assigned only on an arc of C. Methods of finding representations of $f(z)$ in D are given. *A. J. Lohwater* (Ann Arbor, Mich.).

***Mikolás, Miklós. A general integral formula.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 519–540. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian summary)

***Mikolás, Miklós. Sur une extension de la formule d'Euler-Mac-Laurin, se rapportant à des intégrales curvilignes complexes.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 541–550. Akadémiai Kiadó, Budapest, 1952.

The following generalization of the Euler-Maclaurin formula is discussed. Let L be a simple rectifiable arc and $f(z)$ analytic in a domain containing L. We divide L by the points z_ν, $0\leqq\nu\leqq n$. Then the functions $U_\lambda(z)$ can be determined in such a way that the following identity holds:

$$\sum_{k=1}^{n}(z_k-z_{k-1})f(z_k)=\int_L f(z)dz-\int_L U_\nu(z)f^{(\nu)}(z)dz + \sum_{\lambda=1}^{\nu}U_\lambda(z_0)\left[f^{(\lambda-1)}(z_n)-f^{(\lambda-1)}(z_0)\right].$$

Here

$$U_\lambda(z)=(z_n-z_0)^{\lambda-1}\sum_{k=1}^{n}(z_k-z_{k-1})B_\lambda\left(\frac{z_k-z}{z_n-z_0}\right)+G_{\lambda-1}(z)$$

where B_λ is Bernoulli's polynomial and

$G_{\lambda-1}(z)=0$ between z_0 and z_1;

$$G_{\lambda-1}(z)=\frac{1}{(\lambda-1)!}\sum_{k=1}^{\nu}(z_k-z_{k-1})(z_k-z)^{\lambda-1}$$

between z_ν and $z_{\nu+1}$, $1\leqq\nu\leqq n-1$.

The author proves various properties of the functions $U_\lambda(z)$ and investigates their asymptotic behavior when λ is fixed and the subdivision points become dense on L, or when the points are fixed and $\lambda\to\infty$. *G. Szegő* (Stanford, Calif.).

Šmul'yan, Yu. L. Riemann's problem for positive definite matrices. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 143–145 (1953). (Russian)

Let L be the unit circle $|z|=1$, D^+ its interior, D^- its exterior, $A(z)$ a matrix of functions $a_{ij}(z)$ $(i, j=1, \cdots, n)$ defined on L and satisfying a Hölder condition, $\det A\neq 0$ on L. The homogeneous Riemann problem is to find a piecewise continuous vector $\varphi(z)$ with components $\varphi_i^+(z)$ regular in D^+ and $\varphi_i^-(z)$ regular in D^- such that $\varphi^+=A\varphi^-$ on L. The author proves that if A is positive definite on L then the particular indices of this problem are all 0 [for definition of particular indices see Gahov, Doklady Akad. Nauk SSSR (N.S.) 67, 601–604 (1949); these Rev. 11, 169]. This result is used to obtain from the solution of the Riemann problem the solution to the following problem: To find a matrix $F(z)$ of functions $f_{ij}(z)$ regular in D^+ and continuous in $|z|\leqq 1$ such that $\det F(z)\neq 0$ for $|z|\leqq 1$ and $FF^*=A$ on L.

M. Golomb (Lafayette, Ind.).

Ponting, F. W. The location of singularities on the circle of convergence of gap series. I. Quart. J. Math., Oxford Ser. (2) 4, 19–35 (1953).

Extension of some results of Macintyre and Wilson [Proc. London Math. Soc. (2) 47, 60–80 (1940); these Rev. 2, 274] concerning the distribution of singularities, supposed isolated, non-critical, and of finite exponential order, on the circle of convergence, when the upper density of "small" coefficients is given. *S. Mandelbrojt* (Houston, Tex.).

Sunyer i Balaguer, Ferran. Sur les directions de Borel-Valiron communes à une fonction entière, à ses dérivées et à ses intégrales successives. C. R. Acad. Sci. Paris 236, 2196–2198 (1953).

It is asserted that the results of Milloux [J. Analyse Math. 1, 244–330 (1951); these Rev. 13, 930] hold for "directions of Borel-Valiron of maximum kind" as well as for the complete set of directions of Borel-Valiron considered by Milloux. A theorem concerning directions of Borel-Valiron common to an integral function and all its derivatives and integrals and one concerned with gap power series are stated. *A. J. Macintyre* (Aberdeen).

Boas, R. P., Jr. Some elementary theorems on entire functions. Rend. Circ. Mat. Palermo (2) 1 (1952), 323–331 (1953).

The author exposes a simple method by which he systematically proves a number of (mostly well-known) inequalities for entire functions $\Phi(z)$ of finite order. The following result is typical:

$$\liminf \frac{n(r)}{N(r)} \leqq \liminf \frac{\log n(r)}{\log r} \leqq \rho_1 \leqq \limsup \frac{\log n(r)}{\log r} \leqq \limsup \frac{n(r)}{N(r)},$$

where $n(r)$ denotes the number of zeros of $\Phi(z)$ in $|z| \leqq r$, $N(r) = \int_0^r t^{-1} n(t)\,dt$ (assuming $\Phi(0) = 1$, say), and where ρ_1 is the exponent of convergence of the zeros of $\Phi(z)$. Other inequalities involve $\mu(r)$, the modulus of the maximum term in the power series for $\Phi(z)$ for $|z| = r$, and $\nu(r)$, the index of the maximum term. Most of these inequalities were first proved by Shah [J. Indian Math. Soc. (N.S.) 5, 189–191 (1941); 6, 63–68 (1942); Math. Student 10, 80–82 (1942); 12, 67–70 (1945); these Rev. 4, 6, 137; 6, 263]; some go back to Pólya and Valiron. The author also uses his method to prove that for functions of order 1, for any $\epsilon > 0$,

$$\limsup m(r)\{M(r)\}^{1+\epsilon} = \infty,$$

a result first proved (in a sharper form) by Cartwright [Proc. Cambridge Philos. Soc. 30, 412–420 (1934)].

J. Korevaar (Madison, Wis.).

Bieberbach, Ludwig. Über einen Satz Pólyascher Art. Arch. Math. 4, 23–27 (1953).

We denote the order of an entire function $f(z)$ by ρ and its type by σ. Assuming that $f(z)$ and all its derivatives are integral at the points $z = 0, 1, \cdots, m-1$, one must have either $\rho > m$ or $\rho = m$ and $\sigma \geqq 1$. The example

$$f(z) = \exp\{z(z-1)\cdots(z-m+1)\}$$

has this property and illustrates that the assertion is sharp. The author indicates the connection of this theorem with that of Carlson and Pólya. The proof is given in the author's "Theorie der geometrischen Konstruktionen" [Birkhäuser, Basel, 1952; these Rev. 14, 677]. *G. Szegő.*

***Leont'ev, A. F. Ryady polinomov Dirihlet i ih obobščeniya. [Series of Dirichlet polynomials and their generalizations.]** Trudy Mat. Inst. Steklov., vol. 39. Izdat. Akad. Nauk SSSR, Moscow, 1951. 214 pp. 8.80 rubles.

Soit $f(z)$ une fonction entière, et soit $\{\lambda_n\}$ une suite de nombres complexes, $0 < |\lambda_1| \leqq |\lambda_2| \leqq \cdots \leqq |\lambda_n| \leqq \cdots$. Si la suite $\{f(\lambda_n z)\}$ n'est pas complète dans un domaine D, la convergence uniforme d'une combinaison linéaire (1) $P_n(z) = \sum_1^n \alpha_{nj} f(\lambda_j z)$ dans D (c'est-à-dire, la convergence uniforme dans tout compact intérieur à D) implique des propriétés de la limite (2) $P(z) = \lim P_n(z)$, qu'il s'agit de mettre en évidence. Ainsi, on peut définir un opérateur, dépendant de f et de $\{\lambda_n\}$, annulant toute fonction P définie par (2). L'équation fonctionnelle ainsi établie permet de fournir des propriétés importantes aussi bien de la fonction P que de la suite (1) elle-même. On peut, par exemple, étendre le domaine où (2) a lieu, et où, par conséquent, P est encore holomorphe. Si $f(z) = \sum a_n z^n$, avec $a_n \neq 0$ $(n \geqq 0)$, est d'ordre $\rho > 0$, du type $\sigma \neq 0, \infty$, $\liminf |a_n|^{1/n} n^{1/\rho} = (\sigma e \rho)^{1/\rho} > 0$, à toute fonction entière $L(z)$ du même ordre ρ et du type $\sigma_1 < \infty$ correspond une fonction $\gamma(z, t)$ holomorphe en t, lorsque $|z| \leqq R$, dans un cercle $|t| > \mu(R)$, et qui permet d'écrire $2\pi i L(\zeta) f(\zeta z) = \int \gamma(z, t) f(\zeta t)\,dt$, l'intégrale étant étendue sur un cercle autour de l'origine contenant à l'interieur toutes les singularités de γ par rapport à t. A partir de L on forme l'opérateur $M(F) = (2\pi i)^{-1} \int \gamma(z, t) F(t)\,dt$; lorsque F est holomorphe dans $|z| < R_1$, l'intégrale est étendue sur $|t| = R' < R_1$. Si (3) $\limsup n|\lambda_n|^{-\rho} = l < \infty$, on prend pour L les fonctions $L_{k,m}(z) = \prod_{j=k}^{\infty}(1 - z^m \lambda_j^{-m})$, où m est un entier supérieur à ρ. Soit $M_{k,m}$ l'opérateur correspondant. Les résultats fondamentaux, sont alors les suivants: si (1) converge uniformément dans $\mu(R) = (\sigma_1/\underline{\sigma} + (\sigma/\underline{\sigma}) R^{\rho})^{1/\rho}$, $\sigma_1 = \pi\tau/\sin \pi\rho/m$, P satisfait dans $|z| < R$ à la relation $M_{1,m}(P) = 0$ (le systéme $\{f(\lambda_j z)\}$ n'est donc complet dans aucun cercle $|z| < R'$, avec $R' > \mu(0)$), et les limites $\alpha_{nj} = \alpha_j$ $(j \geqq 1)$ existent. Les α_j définissent d'une manière unique P. D'ailleurs si (2) converge dans $|z| < R'$ avec $R' > \mu(0)$, et si P est holomorphe dans $|z| < \mu(R)$, on a dans $|z| < R$ uniformément $P(z) = \lim Q_k(z)$, où $Q_k(z) = \sum_{j=1}^{k-1} \alpha_j L_{k,m}(\lambda_j) f(\lambda_j z)$. Mais des conditions sont indiquées pour que (4) $\sum \alpha_j f(\lambda_j z)$ converge dans un cercle où P est holomorphe. Citons un des corollaires du théorème obtenu: si $\lim n^{1/\rho} |a_n|^{1/n} = (\sigma e \rho)^{1/\rho}$, (5) $\lim n\lambda_n^{-\rho} = 0$, (6) $\delta = -\limsup |\lambda_n|^{-\rho} \log |F'(\lambda_n)| \leqq 0$, où $F(z) = \prod'(1 - z^m \lambda_n^{-m})$, le produit étant étendu sur tous les λ_n dont les puissances $m$ièmes sont distinctes, et si (1) converge dans un voisinage de l'origine, (4) converge dans tout cercle d'holomorphie autour de l'origine. En posant, en particulier $f(z) = e^{-z}$, en supposant les λ_n réels positifs, on voit que de (5) (avec $\rho = 1$) et (6) (avec $m = 2$) et du fait que $\sum_1^n \alpha_n e^{-\lambda_j s}$ converge uniformément dans un voisinage de z_0 résulte que $\sum \alpha_j e^{-\lambda_j s}$ admet l'axe de convergence comme coupure. Des théorèmes classiques sur les séries admettant l'axe de convergence comme coupure en sont des cas particuliers. Il est essentiel de remarquer que déjà (3), avec $\rho = 1$, implique la possibilité du prolongement de l'opérateur à tout canal de largeur supérieure à $2\pi\tau$ (domaine balayé par un disque de rayon supérieur à $\pi\tau$), où P peut être prolongée à partir de D. Si $D \equiv (|z - z_0| < r)$, $r > \pi\tau$, le canal $D(z_0)$ (de largeur supérieure à $2\pi\tau$) où P peut être prolongée est simplement connexe et a un seul feuillet. Dans $D(z_0)$, $M_{k,m}(P)$ a un sens et $\lim M_{k,m}(P) = P$. L'auteur énonce aussi le résultat suivant: si $\tau = 0$, $\rho = 1$, le domaine d'existence de P est convexe, sans citer à cette occasion Pólya, qui a démontré ce théorème antérieurement [Nachr. Ges. Wiss.

Göttingen. Math.-Phys. Kl. 1927, 187–195]. On particularise les λ_n, tout en les supposant complexes, pour obtenir des théorèmes de plus en plus précis. On considère, par exemple, le cas où les λ_n sont les zéros d'une fonction entière, paire, de premier ordre, de type fini, dont l'indicatrice de croissance est $h(\phi)=\sigma|\sin\phi|$. Les résultats obtenus permettent de traiter plusieurs autres problèmes, par exemple, celui qui consiste à indiquer des conditions sur les a_n pour qu'il existe une fonction entière d'ordre non supérieur à ρ, du type fini, qui, aux points λ_n, avec $\limsup n\lambda_n^{-\rho}<\infty$, prenne les valeurs a_n. On termine le livre par la recherche de la solution de certaines équations de la forme

$$\sum_{m=0}^{p}\sum_{n=0}^{q}a_{m,n}f^{(m)}(z+h_n)=0.$$

Les opérateurs introduits au début de ce travail sont alors d'un grand secours. Les méthodes sont intéressantes; les résultats importants. La bibliographie n'est pas toujours respectée. Ainsi, par exemple, on ne trouve aucune mention des travaux de Valiron, Ritt, Hille qui ont été les premiers à signaler l'importance de la convexité dans l'étude des séries de Dirichlet à exposant complexes, et qui ont établi des résultats dont quelques-uns sont démontrés, sous une forme plus générale il est vrai, dans le livre actuel [voir Valiron, Ann. Sci. Ecole Norm. Sup. (3) 46, 25–53 (1929)]. Le théorème fondamental de la thèse de Schwartz ($\sum\lambda_j^{-1}<\infty$) [Etude des sommes d'exponentielles réelles, Hermann, Paris, 1943; ces Rev. 7, 294] est signalé dans la préface de la monographie comme étant dû à l'auteur. [On cite en effet, à ce sujet Leont'ev, Doklady Akad. Nauk SSSR (N.S.) 72, 621–624 (1950); ces Rev. 12, 21.]

S. Mandelbrojt (Houston, Tex.).

Mandelbrojt, Szolem. Sur les propriétés arithmétiques des exposants d'une série de Dirichlet. C. R. Acad. Sci. Paris 236, 1464–1466 (1953).

Let $f(s)$ be an analytic function, uniform in its domain of existence, having the representation

$$f(s)=\sum a_n\exp(-\lambda_n s)\quad(0<\lambda_n\uparrow\infty)$$

for $\sigma>\sigma_c=0$ ($s=\sigma+it$). Let $a<0$ and denote by R_a the largest domain containing the half-plane $\sigma>0$ while contained in the intersection of the domain of existence of $f(s)$ and the half-plane $\sigma>a$. The author announces a number of results relating the arithmetic properties of the exponents with the singularities and the growth properties of the function of which the following is a typical example. Let $(\lambda_n)=\lambda_n-[\lambda_n]$ and denote by $\nu(\alpha)$ the number of times that α is taken by $\{(\lambda_n)\}$ and by $\lambda(\alpha)$ the smallest λ_n such that $(\lambda_n)=\alpha$. If the following holds: (i) $\liminf(\lambda_{n+1}-\lambda_n)>0$; (ii) $\lim(\log a_n/\lambda_n)=0$; (iii) $f(s)$ is bounded except for singularities in $\sigma>a$ ($a<0$) (that is, $f(s)$ is bounded in any subset of R_a having a positive distance from the boundary of R_a); (iv) there exists an infinite sequence $\{\alpha_p\}$ of isolated points of $\{(\lambda_n)\}$ such that $\lim\nu(\alpha_p)/\lambda(\alpha_p)=0$ ($p\to\infty$); then, the axis of convergence is a cut modulo $2\pi i$ for the series in the sense that given $s_0=it_0$ and $\epsilon>0$, there exists a point s not belonging to R_a such that $(s-s_0)/2\pi i$ differs by less than ϵ from an integer.

S. Agmon (Jerusalem).

Leja, F. Sur une famille de fonctions analytiques extrémales. Ann. Soc. Polon. Math. 25 (1952), 1–16 (1953).

Let D be a domain, whose frontier F consists of an outer curve C_0 and inner curves C_k, $k=1,\cdots,\nu$. Let D_k be the interior of C_k and a_k a point of D_k for $k=1,\cdots,\nu$. Let $A(z)$ be continuous and positive for z on F. By a procedure involving the taking of real powers, moduli, and sums starting from certain polynomials, the author constructs a sequence of functions, whose limit $\Phi(z,\lambda)$ has the following properties: (i) $\Phi(z,\lambda)$ exists positively for $z\neq a_1,a_2,\cdots,a_k$; (ii) $\log\Phi(z,\lambda)$ is harmonic except on a closed subset F_λ of F where $\Phi(z,\lambda)=[A(z)]^\lambda$;

(iii) $$\log\Phi(z,\lambda)=\sigma_k\log(1/(z-a_k))+O(1)$$

near a_k, where σ_k is a preassigned constant; (iv) when $\lambda=0$, $F_\lambda=F$, so that $\log\Phi(z,0)=\sigma_k g(z,D_k)$ for z in D_k, where $g(z,D_k)$ is the Green's function of D_k at a_k.

W. K. Hayman (Exeter).

Komatu, Yûsaku, and Nagura, Shôhei. Theory of univalent functions. Sûgaku 1, 286–302 (1949). (Japanese)

The authors first give a simple proof to a theorem of A. C. Schaeffer and D. C. Spencer [Duke Math. J. 12, 107–125 (1945); these Rev. 6, 206] concerning the fundamental differential equation of Löwner's type for schlicht conformal map of a unit disc. Then various distortion and rotation theorems are derived from this result. Most of the results of this paper can be found in two papers (in English) of the authors [Y. Komatu, Proc. Japan Acad. 25, no. 1, 1–10 (1949); these Rev. 12, 490; S. Nagura and Y. Komatu, Nagoya Math. J. 1, 25–33 (1950); these Rev. 12, 490].

S. Kakutani (New Haven, Conn.).

Zmorovič, V. A. On some classes of analytic functions univalent in a circular ring. Mat. Sbornik N.S. 32(74), 633–652 (1953). (Russian)

Let $f(z)$ be regular in a ring $K:0<q<|z|<1$, and normalized by the condition that $\int f(z)z^{-1}dz=2\pi i$ on $q<r=|z|<1$. It is proved that a necessary and sufficient condition that $\Re(f(z))>0$ in K is that

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(ze^{-i\theta})d\mu_1(\theta)+\frac{1}{2\pi}\int_{-\pi}^{\pi}F\left(\frac{q}{z}e^{i\theta}\right)d\mu_2(\theta)-1,$$

where

$$F(z)=\frac{1+z}{1-z}+2\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}(z^k-z^{-k}),$$

and $\mu_j(\theta)$, $j=1,2$, is a nondecreasing function on $[\pi,-\pi]$ for which $\mu_j(-\pi+0)=\mu_j(-\pi)=0$ and $\mu_j(\pi)=2\pi$.

This theorem contains a variety of previously known results as special cases. Application is made to obtain formulas for functions which are regular and univalent in K and (a) convex, (b) starlike, and also for functions meromorphic and univalent in K.

A. W. Goodman.

Gagua, M. B. On a theorem of Hardy and Littlewood. Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 121–125 (1953). (Russian)

A function $f(z)$ is said to be of class Λ_p on a set E if $|f(z_1)-f(z_2)|\leq A|\log|z_1-z_2|^{-1}|^{-p}$ for $z_1, z_2\in E$. Let T be a bounded simply connected region with boundary L which satisfies the conditions $|z(w_1)-z(w_2)|\leq M|w_1-w_2|^{\alpha_1}$, $|w_1|=|w_2|=1$; $|w(z_1)-w(z_2)|\leq M_2|z_1-z_2|^{\alpha_2}$, $z_1, z_2\in L$; $0<\alpha_i\leq 1$, where $z(w)$ maps the unit circle conformally onto T and $w(z)$ is its inverse. The author proves that if f is analytic in T, continuous in $T+L$, and of class Λ_p on L, then it is Λ_p in $T+L$. This extends theorems of Hardy and Littlewood and of Walsh and Sewell. Applications are made to Cauchy integrals and to the solutions of linear partial differential equations with entire coefficients.

P. Davis.

Lenz, Hanfried. Über Verallgemeinerungen der Schwarzschen Polygonabbildung. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 13–29 (1953).

The author generalizes Schwarz' result on conformal mapping of simply-connected polygons, rectilinear or circular, to more general domains, especially to multiply-connected polygons. A differential equation of the third order is valid for any mapping function of a Riemann half-surface onto a polygon. According to special configurations, the equation may be of lower order. Various cases are enumerated in view of this classification. *Y. Komatu* (Tokyo).

Stallmann, Friedemann. Zum Parameterproblem der konformen Abbildung von Kreisbogenpolygonen. Mitt. Math. Sem. Univ. Giessen no. **43**, i+46 pp. (1952).

This paper deals with the old and tricky problem as to the connection between the accessory parameters of a Fuchsian differential equation and the geometric type of the circular arc polygon whose conformal map onto a half-plane is determined by the ratio of two solutions. By using an approximate solution of the equation which is valid for large values of the accessory parameters, the author brings some order into the chaos and arrives at approximate polygonal shapes which are the more reliable the larger the parameters are. *Z. Nehari* (St. Louis, Mo.).

Strebel, Kurt. Über die konforme Abbildung von Gebieten unendlich hohen Zusammenhangs. I. Comment. Math. Helv. **27**, 101–127 (1953).

The author gives an analysis of the canonical conformal mapping of plane domains of infinite connectivity based on the behavior of an extremal distance from the boundary which is defined in terms of extremal length.

P. R. Garabedian (Stanford, Calif.).

Röhrl, Helmut. Über gewisse Verallgemeinerungen der Abelschen Integrale. Math. Nachr. **9**, 23–44 (1953).

Questo lavoro è strettamente collegato con altri dello stesso autore [questi Rev. **13**, 224, 637, 736; **14**, 462], nonchè con lavori di altri autori; l'argomento è lo studio degli integrali sopra una superficie di Riemann $\mathfrak{F}$, analoghi agli integrali abeliani, portanti su funzioni del tipo $A(z)\exp J(z)$, essendo $A(z)$ una funzione algebrica appartenente ad $\mathfrak{F}$ e $J(z)$ un arbitrario integrale abeliano.

Ecco i principali risultati. Sia k la classe generata da $J(z)$, cioè l'insieme delle funzioni $A(z)\exp J(z)$, con A variabile e J fisso. Definiti per k i possibili cammini d'integrazione e i cosidetti cammini chiusi, si costruisce una particolare base lineare di cammini $\mathfrak{C}_\sigma$; si costruisce inoltre una base di funzioni $\mathfrak{E}^{(\nu)}$ per la classe k, anch'essa opportunamente ordinata. Allora gli integrali

$$p_{\mu\nu}=\int_{\mathfrak{C}_\mu}\mathfrak{E}^{(\nu)}dz$$

formano una matrice di periodi

$$\mathfrak{P}=2\pi i\left\|\begin{matrix} U & 0 \\ * & \mathfrak{P}_0 \end{matrix}\right\|.$$

Costruita, in relazione alla prima, una base per la classe $\check{k}$, complementare di k, cioè generata da $-J(z)$, si trovano le relazioni di Weierstrass, che esprimono come forme bilineari negli elementi di $\mathfrak{P}_0$ e $\check{\mathfrak{P}}_0$ gli integrali

$$\int_{\mathfrak{C}_\sigma}\int_{\check{\mathfrak{C}}_\tau}[E^{(\nu)}(z,x)dx-d\check{F}^{(\nu)}(x,z)]dz$$

ove $E^{(\nu)}$, $\check{F}^{(\nu)}$ sono funzioni collegate con la base scelta. Seguono i teoremi di scambio, che esprimono una simmetria formale di certe forme differenziali, la decomposizione di ogni funzione della classe in funzioni prime del tipo di Prym e Weierstrass, e il teorema di Abel, nelle due forme aritmetica e trascendente, che dà le condizioni affinchè in una data classe k esista una funzione divisibile per un prefissato divisore, evente per ordine un certo intero caratteristico della classe, detto genere relativo. *F. Conforto* (Roma).

Stoïlow, S. Note sur les fonctions analytiques multiformes. Ann. Soc. Polon. Math. **25** (1952), 69–74 (1953).

The author introduces the following class I of Riemann surfaces. Let R be a covering surface of the Riemann sphere S, defined by a 2-dimensional manifold V and a mapping $z=T(p)$ of V. Then R is said to be of class I if it has the Iversen property: Given a path l on S from a to b, there exists for any $\alpha\in T^{-1}(a)$ a path λ on V from α to β, such that $T(\lambda)$ lies in an arbitrarily given neighborhood of l, and $b'=T(\beta)$ is arbitrarily close to b.

Replacing S by another Riemann surface R_0, defined by V_0 and $z=T_0(p_0)$, $p_0\in V_0$, a Riemann surface R of class I on R_0 is defined in an analogous manner for $p_0=T(p)$, $p\in V$. The author lists a number of covering surfaces which are of class I, e.g., those corresponding to Abelian integrals on R_0.

The main significance of the class I lies in its relation to boundary components Γ of R, defined by decreasing nested sequences of noncompact regions $D_i\subset R$ [Stoïlow, Leçons sur les principes topologiques de la théorie des fonctions analytiques, Gauthier-Villars, Paris, 1937, p. 86]. Let $\overline{T(D_i)}$ be the closure of the set $T(D_i)$ on R_0, and set $C=\bigcap\overline{T(D_i)}$. The author calls Γ punctual, exterior to R_0, or totally spread over R_0, according as C reduces to a point, is empty, or covers all of R_0, respectively. In other cases Γ is said to be partially spread. The following theorem is given: If R is of class I on R_0, then either (1) each Γ is punctual or exterior to R_0, and R consists of a finite number of sheets over R_0, or else (2) there exists at least one Γ totally spread over R_0, and the Γ which are not of this kind are punctual or exterior to R_0; in this case, the number of sheets of R is infinite.

The author shows that, given a covering surface R of S, there exists a region B of S, such that a proper part of R is of class I on B, while this is not always true for all of R.

L. Sario (Cambridge, Mass.).

Flathe, Herbert. Approximation analytischer Funktionen auf nichtgeschlossenen Riemannschen Flächen. Math. Ann. **125** (1952), 287–306 (1953).

$\mathfrak{R}$ sei eine geschlossene Riemann'sche Fläche, $\mathfrak{B}$ ein Teilbereich von $\mathfrak{R}$. Eine in $\mathfrak{B}$ eindeutige reguläre Funktion $f(z)$ ist dann und nur dann durch in ganz $\mathfrak{R}$ eindeutige reguläre Funktionen gleichmässig in $\mathfrak{B}$ approximierbar, falls $\mathfrak{B}$ zu $\mathfrak{R}$ einfach zusammenhängend ist [siehe Behnke und Stein, Math. Ann. **120**, 430–461 (1949); diese Rev. **10**, 696]. Während dieser Satz von Behnke-Stein eine reine Existenzaussage ist, gibt Verfasser für in speziellen Teilbereichen reguläre Funktionen ein Verfahren zur Konstruktion von in ganz $\mathfrak{R}$ noch regulären Approximationsfunktionen an. Es handelt sich um Bereiche, welche bei geeigneter analytischer Transformation von $\mathfrak{R}$ in mehrblättrige Kreise übergehen und um eine weitere Verallgemeinerung solcher Bereiche.

P. Thullen (Genf).

Borel, Armand. Les fonctions automorphes de plusieurs variables complexes. Bull. Soc. Math. France **80**, 167–182 (1952).

This is a brief but, in a certain sense, comprehensive survey of automorphic functions in n complex variables, and frequently the author makes a statement for a bounded domain, when, axiomatically at least, an extension to a complex space would be readily available, and (dealing with domains, as he does) the author envisages only the traditional automorphy factor $[\partial(y_1, \cdots, y_n)/\partial(x_1, \cdots, x_n)]^m$, without questioning its uniqueness, as was done so successfully by Appell in connection with the Jacobi automorphy factor for multi-periodic functions in n variables.

The first three sections of the paper deal with (i) conditions for the existence of n functionally independent meromorphic automorphic functions of sufficiently high weight, (ii) algebraic relation between $n+1$ meromorphic automorphic functions, (iii) isobaric relations between $n+2$ holomorphic automorphic forms, without taking notice of a paper by the reviewer [J. Indian Math. Soc. (N.S.) **16**, 1–6 (1952); these Rev. **13**, 932] in which such isobaric relations have been established, almost trivially, on complex spaces of a very general kind, and (iv) with the more vexing problem of existence and dependence of "modular" functions in case the fundamental domain of automorphy is not a compact one.

Finally, in section IV, as a background to automorphic functions in matrix spaces, the author announces new results from the ever-fascinating topic of the structure of complex symmetric spaces which will be presented in full in a forthcoming joint paper with A. Lichnerowicz. These results are as yet too tersely stated to be described, but they have made this reviewer feel as keenly as ever that somebody ought to write a full-length book on real and complex symmetric spaces. *S. Bochner* (Princeton, N. J.).

***Severi, Francesco. Funzioni analitiche e forme differenziali.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 125–140. Casa Editrice Perrella, Roma, 1953.

The author surveys nicely some residue formulas due to Poincaré-Picard, Gauss, Wirtinger, and Didon-Caccioppoli, and with regard to the last one he states the following theorem in two variables, which is extensible to more. If f, g are meromorphic in the complex variables x, y in a bounded domain D; if F, G are their zero-varieties and F_1, G_1 their pole-varieties, with no points of indeterminacy existing; if S_1, in suitable reckoning, is the number of joint points of (F, F_1), S_2 of (G, G_1), and S_3 of (F, G_1) plus (F_1, G); then $S_1+S_2-S_3$ can be expressed as an integral

$$-\frac{1}{4\pi^2}\int_\sigma \frac{1}{fg}\frac{\partial(f,g)}{\partial(x,y)}dxdy$$

in which σ is a 2-cycle in $D-F-G-F_1-G_1$ which is homologous to a number of small toroidal cycles each of which is, in a certain sense, "simply" and "positively" contractible on one of the points mentioned before.

S. Bochner (Princeton, N. J.).

Theory of Series

***Chandrasekharan, K., and Minakshisundaram, S. Typical means.** Oxford University Press, 1952. x+139 pp. $6.50.

Le petit livre de Hardy et Riesz [The general theory of Dirichlet series, Cambridge, 1915] épuisé depuis de nombreuses années, était consacré aux moyennes typiques de Marcel Riesz et à leurs applications à la série de Dirichlet. La monographie actuelle, en reprenant le sujet et en résumant d'une manière systématique et concise les progrès essentiels faits depuis dans cette théorie, vaste et parsemée de résultats d'importance inégale, comble heureusement une lacune. Ceux qui se souviennent avec nostalgie de leur première lecture du très beau Cambridge Tract de Hardy et Riesz, ainsi que les lecteurs, plus jeunes, qui s'initient au domaine, en sauront gré aux auteurs. Les suites $\{a_n\}$, $\{\lambda_n\}$, avec $0<\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty$ étant données, et en posant $A(t)=\sum_{\lambda_\nu\le t}a_\nu$, $A^k(t)=\sum_{\lambda_\nu\le t}(t-\lambda_\nu)^k a_\nu = k\int_0^t(t-\tau)^{k-1}A(\tau)d\tau$, $C^k(x)=x^{-k}A^k(x)$, on sait que la série $\sum a_\nu$ est dite sommable par les moyennes typiques d'ordre k et du type λ, ou sommable (λ, k), si $C^k(x)\to s$ (fini), avec $x\to\infty$. La série $\sum a_\nu$ est bornée (λ, k) si $C^k(x)=O(1)$. Dans le chapitre I, après avoir démontre que la sommabilité (λ, k) entraîne la sommabilité (λ, k') avec $k'>k$, et, après avoir indiqué des relations entre l'ordre de grandeur des quantités $A^r(x)$ et $A^k(x)$, pour $r<k$, les auteurs démontrent le théorème fondamental de Riesz constituant, en un certain sens, une relation de convexité entre (les logarithms de) $|A(x)|$, $|A^r(x)|$ et $|A^k(x)|$. Ce théorème conduit aux théorèmes taubériens. Le chapitre II est consacré aux théorèmes qui résument essentiellement le fait suivant: k restant fixe, la puissance de la sommabilité augmente avec la "fréquence" des λ_n. D'une manière plus générale, des conditions sont indiquées portant sur $\mu(\lambda)$ pour que de la sommabilité (λ, k) (ou seulement du fait que la série est bornée (λ, k)) résulte la sommabilité (μ, k). Le chapitre III est consacré aux applications des moyennes typiques à la série de Dirichlet: théorèmes du type abélien, sommabilité uniforme dans un angle, abscisses de sommabilité, ordre du prolongement analytique dans le demi-plan de sommabilité (λ, k), relations entre les sommabilités (λ, k) et (l, k) où $l_n=e^{\lambda_n}$, etc.; sommabilité aux points réguliers sur l'axe de la sommabilité, et, enfin, généralisations à la sommabilité du Taubérien classique concernant les séries de Taylor ("reciproques" des théorèmes abéliens). Une partie du chapitre est consacrée aux produits de Dirichlet des séries sommables. Le chapitre IV s'occupe des applications des moyennes typiques aux séries de Fourier de plusieurs variables. La formule de Bochner liant les moyennes de Riesz d'une série de Fourier de k variables à la moyenne sphérique de la fonction correspondante permet de déduire la sommabilité de la série par les moyennes d'un certain ordre δ (dépendant seulement de k) en tout point où la fonction est continue. Si l'ordre des coefficients est convenablement limité, on peut diminuer δ. On étudie alors la sommabilité des séries qui dérivent d'une série de Fourier par l'application repétée de l'opérateur différentiel de Laplace ("laplacien"). La fin du chapitre est consacrée à l'établissement des conditions nécessaires et suffisantes pour la sommabilité (et la sommabilité absolue) d'une série de Fourier en un point. On définit, pour le faire, des moyennes sphériques d'ordre supérieur d'une fonction (la moyenne sphérique ordinaire étant considerée comme étant d'ordre zéro), et on généralise la formule de Bochner. Chaque chapitre est suivi d'une notice bibliographique et historique.

S. Mandelbrojt (Houston, Tex.).

San Juan, Ricardo. Les fondements d'une théorie générale des séries divergentes. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 45–76 (1951).

[This is a summary of a paper which is to appear in Revista Acad. Ci. Madrid.] Let $f(z)$ have the asymptotic series $\sum a_\nu z^\nu$ in a domain D, with bounds m_n, so that

$|f(z)-\sum_{n=0}^{n_0-1}a_nz^n||z|^{-n}<m_n$ for $n>n_0$. Let $\{m_n\}$ satisfy one of the equivalent divergence conditions which are known to ensure that two functions with the same a_n and the same m_n in D are identical. If D is a curvilinear sector $S(\alpha, z)=\{|z^{1/n}-z_0^{1/n}|<|z_0|^{1/n}\}$ one of these conditions is $(C_n):\sum(\bar{m}_n^{1/n})^{1/n}=\infty$, where $\bar{m}_n$ is the convex regularization of m_n. In this case it may still be true that there are different functions with the same a_n but different m_n. However, this cannot occur if, among all the $f(z)$ with the same a_n, there is one for which the bounds m_n are smallest, and in this case the author calls the corresponding $f(z)$ a function of optimum asymptotic approximation. The approximation is called regular if $\{m_n\}$ is nondecreasing. The author calls $f(z)$ the radial continuation of $\sum a_nz^n$ along the vector $0z$ if it is the function of regular optimum approximation for the series in an $S(\alpha, z)$ and also in every interior $S(\alpha', z)$ in which there is an approximation of the same kind satisfying $(C_{\alpha'})$. He gives some sufficient conditions for an $f(z)$ to be a radial continuation; further conditions are given in the paper reviewed below.

The author also describes some results on the decomposition of a given function into the sum of two functions belonging to different quasi-analytic classes, and on the construction of functions of the form $\sum A_n/(1+\eta_nz)$, not identically 0, but with all derivatives zero at the origin. By means of a Laplace transformation these lead to some examples of non-uniqueness for asymptotic power series. *R. P. Boas, Jr.*

San Juan, Ricardo. Sufficient conditions for radial continuation. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 185–195 (1952). (Spanish)

The author gives some conditions, less restrictive than those of the paper reviewed above, which are sufficient for a function to be the radial continuation (in the sense of the preceding review) of a given power series.

R. P. Boas, Jr. (Evanston, Ill.).

Clunie, James. An extension of quasi-monotone series. Math. Student **20** (1952), 107–112 (1953).

Let $\sum a_n$ be a series of positive terms for which

$$(*) \qquad a_{n+1}\leqq a_n\{1+\varphi(n)/n\}.$$

If the sequence $\varphi(n)$ is bounded then the series $\sum a_n$ is quasi-monotone. Szász [Amer. J. Math. **70**, 203–206 (1948); these Rev. **9**, 278] and Shah [Math. Student **15**, 19–24 (1948); these Rev. **10**, 446] extended several theorems on monotone series to cover quasi-monotone series. For example, if $\sum a_n<\infty$ and $\sum a_n$ is quasi-monotone, then $na_n\to 0$. The present paper shows that if $\varphi(n)\uparrow\infty$ as $n\to\infty$, then the theorems of Szász and Shah cannot be extended from the class of quasi-monotone series to the class of series satisfying (*). *R. P. Agnew* (Ithaca, N. Y.).

Ščeglov, M. P. On subsequences of the arithmetic mean sums of Cesàro. Doklady Akad. Nauk SSSR (N.S.) **87**, 517–520 (1952). (Russian)

Let d and D be the limits of indeterminacy of the arithmetic means σ_n of the partial sums of the series $\sum_0^\infty a_n$ and let d' and D' (not necessarily finite) denote the same for a subsequence σ_{n_m}, so that $d\leqq d'\leqq D'\leqq D$. It is shown that $d=d'$ and $D=D'$ if $a_n<o(n^{-1})$ and $n_{m+1}=O(n_m)$ or if $a_n<O(n^{-1})$ and $n_{m+1}=n_m+o(n_m)$. The second in each pair of conditions cannot be weakened. It is remarked that, in view of the Hardy-Landau Tauberian theorem, if under either of the above sets of conditions σ_{n_m} converges to a finite limit s then $\sum_0^\infty a_n=s$. *G. Klein.*

Ščeglov, M. P. Generalization of the Hardy-Landau-Vijayaraghavan theorem. Doklady Akad. Nauk SSSR (N.S.) **87**, 697–700 (1952). (Russian)

Let d and D be the limits of indeterminacy of the partial sums of the numerical series $\sum_0^\infty a_n$ and let d' and D' be the same for their arithmetic means, so that

$$(1) \qquad d\leqq d'\leqq D'\leqq D.$$

The classical result of the title states that $d=D$ whenever $d'=D'=S$ (finite) if $a_n=O(n^{-1})$. This last condition can be replaced by $a_n<O(n^{-1})$ when S is finite but, as shown by Vijayaraghavan [J. London Math. Soc. **2**, 215–222 (1927)], if $S=+\infty$ the weakest effective one-sided Tauberian condition is $a_n<O(n\log\log n)^{-1}$. Here the author considers the more general situation wherein d' and D' need not be equal or finite. In addition to all the above Tauberian conditions, those obtained by use of o in place of O are studied to determine which of the possibilities (1) do and which cannot occur. For example if $a_n<O(n\log\log n)^{-1}$ then $d=d'\leqq D'=D$ unless $D'=D=+\infty$ and $-\infty<d<d'<+\infty$, but if $a_n<o(n\log\log n)^{-1}$ this alternative is excluded. Some of the results are the best of their kind. *G. Klein.*

Hill, J. D. Summability methods defined by Riemann sums. Canadian J. Math. **5**, 289–296 (1953).

Let $f(x)$ be real and Riemann integrable over $0\leqq x\leqq 1$ with $\int_0^1 f(x)dx=1$. Let δ denote a pattern of points x_k^n such that, for each $n=1, 2, \cdots, (k-1)/n\leqq x_k^{(n)}\leqq k/n$ when $k=1, 2, \cdots, n$. A sequence $s_1, s_2, \cdots$ is said to be evaluable (R, f, δ) to s if $T_n\to s$ as $n\to\infty$ when $T_n=\sum_{k=1}^n f(x_k^n)s_k$. These methods (R, f, δ) are compared with the Cesàro methods (C, α) for evaluating sequences. The following are some of the results obtained. Each bounded sequence evaluable $(C, 1)$ is evaluable (R, f, δ). In order that (R, f, δ) include $(C, 1)$, it is necessary and sufficient that the sequence

$$\sum_{k=1}^{n}(k/n)\,|f(x_k^n)-f(x_{k+1}^n)|$$

be bounded. In order that $f(x)$ be such that (R, f, δ) includes $(C, 1)$ for each fixed δ, it is necessary and sufficient that $xf(x)$ have bounded variation over $0\leqq x\leqq 1$. If $f(x)$ is monotone increasing, then (R, f, δ) is equivalent to $(C, 1)$. If $f(x)$ is monotone decreasing and $f(1)>\frac{1}{2}$, then (R, f, δ) is equivalent to $(C, 1)$. Pertinent examples and remarks are given. *R. P. Agnew* (Ithaca, N. Y.).

Livingston, Arthur E. Some Hausdorff means which exhibit the Gibbs' phenomenon. Pacific J. Math. **3**, 407–415 (1953).

$g(t)$ est définie pour $0\leqq t\leqq 1$, $g(0)=0$, $g(1)=1$, constante par intervalles, les discontinuités ayant pour abscisses $t_1, t_2, \cdots, t_p$. L'auteur précisant et généralisant un résultat de O. Szasz [Trans. Amer. Math. Soc. **69**, 440–456 (1950); ces Rev. **12**, 405], montre que pour la série $\sum \sin kx/x$ à laquelle on applique la méthode de sommation de Hausdorff, le phénomène de Gibbs a lieu si $t_1, \cdots, t_p$ sont linéairement indépendants sur l'ensemble des rationnels de (0, 1) (cette restriction est inutile si $p=2$). *M. Zamansky* (Paris).

Gaier, Dieter, und Peyerimhoff, Alexander. Summierbarkeitsfaktoren bei Eulerschen Reihentransformationen. Math. Z. **58**, 232–242 (1953).

The authors show that the exact region of the z-plane where $\sum a_nz^n$ is C_kE_p-summable for each C_kE_q-summable series $\sum a_n$ (here $k\geqq 0$, $0\leqq p<q$ and E_p is the Euler-Knopp

method of summation) is given by

$$|z|+q(q+1)^{-1}|z-p/q| \leqq (p+1)/(q+1), \quad z \neq -(2q+1)^{-1}.$$

The proof depends upon the Fatou-M. Riesz theorem on the C_k-summability of a power series in a regular point of its circle of convergence. For $k=p=0$ this contains an earlier result by Peyerimhoff [Math. Z. 55, 288–291 (1952); these Rev. 14, 265]. Similar but incomplete results are given for the general Euler method of summation. *G. G. Lorentz.*

Zamansky, Marc. Sur les séries divergentes. C. R. Acad. Sci. Paris **236**, 2291–2293 (1953).

Let A be a class of summability methods of series, $b \circ a$ the superposition of the methods a and b. Let T be a "functional on A connected with the operation $b \circ a$ by some simple algebraic property" such as $T(b \circ a)=T(b)T(a)$. For Hausdorff methods H_g, T is said to be the Mellin transform $\int_0^1 t^{p-1}g(t)dt$ of g. In a way which is not explained, a T satisfying a unicity theorem is said to define for each $a \in A$ a subclass $A_a \subset A$ of methods included in a. As examples, general arithmetic means and Nörlund means are treated. *G. G. Lorentz* (Detroit, Mich.).

Bendukidze, A. D. Strong summation of double numerical series. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 329–334 (1952). (Russian)

Let $p>0$. A double series $\sum u_{ik}$ with partial sums $S_{mn}=\sum_{i=0}^{m}\sum_{k=0}^{n} u_{ik}$ is strongly evaluable with exponent p, or is evaluable H_p, to S if $\lim_{m,n\to\infty} \sigma_{m,n}^{(p)}=0$ where

$$\sigma_{m,n}^{(p)}=(m+1)^{-1}(n+1)^{-1}\sum_{i=0}^{m}\sum_{k=0}^{n}|S_{ik}-S|^p.$$

The series $\sum u_{ik}$ is restrictedly strongly evaluable with exponent p, or is evaluable $H_p^{(\lambda)}$, to S if, for each $\lambda>1$, $\sigma_{m,n}^{(p)}\to 0$ when $m, n\to\infty$ subject to the restriction $\lambda^{-1}<m/n<\lambda$. If $\sum u_{ik}$ is evaluable H_p [or $H_p^{(\lambda)}$] to S and $q<p$, then $\sum u_{ik}$ is evaluable H_q [or $H_q^{(\lambda)}$] to S. If $\sum u_{ik}$ has bounded partial sums and is evaluable H_p [or $H_p^{(\lambda)}$] to S and if $q>p$, then $\sum u_{ik}$ is evaluable H_q [or $H_q^{(\lambda)}$] to S. If $\sum u_{ik}$ converges to S, if $p>0$, if $S_{mn}=o(m+1)^{1/p}$ as $m\to\infty$ when $n=0, 1, 2, \cdots$ and $S_{mn}=o(n+1)^{1/p}$ as $n\to\infty$ for each $m=0, 1, 2, \cdots$, then $\sum u_{ik}$ is evaluable $H_p^{(\lambda)}$ to S. *R. P. Agnew.*

Žak, I. E. On Riemann summability of double numerical series. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 587–593 (1952). (Russian)

A double series $\sum a_{m,n}$ is evaluable to L by the Riemann method $R(2, 2)$ if $\lim_{x,y\to 0} f(x, y)=L$ where

$$f(x, y)=\sum_{m,n=0}^{\infty}\left(\frac{\sin mx}{mx}\right)^2\left(\frac{\sin ny}{ny}\right)^2 a_{m,n},$$

and is restrictedly evaluable to L if, for each $\lambda>1$, $f(x, y)\to L$ as $x, y\to 0$ subject to the restriction $\lambda^{-1}<|x/y|<\lambda$. Some convergent series $\sum a_{mn}$ having unbounded partial sums S_{mn} fail to be evaluable $R(2, 2)$. It is shown that if $\sum a_{mn}$ converges to S and if constants K, α, β exist such that $0<\alpha, \beta<1$ and $|S_{mn}|<K(m+1)^{\alpha}(n+1)^{\beta}$, then $\sum a_{mn}$ is restrictedly evaluable $R(2, 2)$ to S. *R. P. Agnew.*

Kalašnikov, M. D. On conditions of summability of infinite products. Ukrain. Mat. Žurnal **3**, 477–488 (1951). (Russian)

Without using the terminology, the author gives three theorems on regularity of matrix methods for evaluation of infinite products. A matrix a_{nk} determines a regular method for evaluation of infinite products $\prod(1+u_k)$ if $\prod_{k=1}^{n}(1+a_{nk}u_k)$ converges for each $n=1, 2, 3, \cdots$ to a number P_n and $\lim P_n=P$ whenever $\prod(1+u_k)$ converges to $P\neq 0$. A matrix a_{nk} for which $\lim_{k\to\infty} a_{nk}=0$ for each $n=1, 2, 3, \cdots$ is regular if and only if $\lim_{n\to\infty} a_{nk}=1$ for each $k=1, 2, 3, \cdots$ and there exist $2p+1$ sequences $k_n^{(1)}, k_n^{(2)}, \cdots, k_n^{(2p+1)}$ such that $k_n^{(1)}\leqq k_n^{(2)}\leqq k_n^{(3)}\leqq\cdots\leqq k_n^{(2p+1)}$ for each n, $\lim_{n\to\infty} k_n^{(1)}=\infty$, and, for some constant M,

$$\sum_{k=1}^{k_n^{(1)}}|a_{nk}-1|+\sum_{k=k_n^{(1)}+1}^{k_n^{(2)}}|a_{nk}|+\sum_{k=k_n^{(2)}+1}^{k_n^{(3)}}|a_{nk}-1|+\cdots+\sum_{k=k_n^{(2p+1)}+1}^{\infty}|a_{nk}|\leqq M$$

for each $n=1, 2, 3, \cdots$. A matrix a'_{nk} for which $\lim_{k\to\infty} a'_{nk}=1$ for each $n=1, 2, 3, \cdots$ is regular if and only if $\lim_{n\to\infty} a_{nk}=1$ for each $k=1, 2, 3, \cdots$ and there exist $2q$ sequences $k_n^{(1)}, k_n^{(2)}, \cdots, k_n^{(2q)}$ such that $k_n^{(1)}\leqq k_n^{(2)}\leqq\cdots\leqq k_n^{(2q)}$ and, for some constant M,

$$\sum_{k=1}^{k_n^{(1)}}|a_{nk}-1|+\sum_{k=k_n^{(1)}+1}^{k_n^{(2)}}|a_{nk}|+\sum_{k=k_n^{(2)}+1}^{k_n^{(3)}}|a_{nk}-1|+\cdots+\sum_{k=k_n^{(2q)}+1}^{\infty}|a_{nk}-1|\leqq M$$

for each $n=1, 2, 3, \cdots$. Each regular matrix is formed by taking none, some, or all of the rows of a regular matrix of the first type and interspersing none, some, or all of the rows of a regular matrix of the second type. The first theorem and further arguments are used to prove the theorem of M. Schweitzer [Acta Univ. Szeged. Sect. Sci. Math. **11**, 139–146 (1947); these Rev. **9**, 87] that integers $n_1\leqq n_2\leqq n_3\leqq\cdots$ are such that

$$\text{(i)} \qquad \lim_{x\to 1}\prod_{k=1}^{\infty}(1+a_k x^{n_k})=\prod_{k=1}^{\infty}(1+a_k)$$

whenever $\prod(1+a_k)$ converges to a number not zero if and only if (ii) $n_1+n_2+\cdots+n_k=O(n_k)$.

Erratum: In the reference to G. M. Robison [Amer. J. Math. **51**, 653–660 (1929)] the name of Robison is given as Robinson. *R. P. Agnew* (Ithaca, N. Y.).

Gnanadoss, A. A. Periodically recurring series. Math. Student **20** (1952), 119–121 (1953).

The following theorem is proved in this paper. If

$$\text{(1)} \quad u(n+3)=a(n)u(n+2)+b(n)u(n+1)+c(n)u(n),$$

and for all positive integral values of n

$$\text{(2)} \quad a(n+k)=a(n), \quad b(n+k)=b(n), \quad c(n+k)=c(n),$$

where k is a constant positive integer, then (A) the radius of convergence of $\sum_{n=0}^{\infty}u(n)x^n$ is not zero; (B) the generating function $f(x)=\sum_{n=0}^{\infty}u(n)x^n$ is rational; (C) there is a scale of relation of the type $1-px^k-qr^{2k}-rx^{3k}$ where p, q, r are constants. *T. Fort* (Athens, Ga.).

Lane, Ralph E. Absolute convergence of continued fractions. Proc. Amer. Math. Soc. **3**, 904–913 (1952).

A continued fraction

$$\text{(1)} \qquad b_0+a_1/b_1-a_2/b_2-\cdots-a_n/b_n-\cdots,$$

where b_n, a_{n+1} are complex numbers and $a_{n+1}\neq 0$, $n=0, 1, \cdots$, converges absolutely if the series equivalent to the sequence of approximants $\{f_n\}$ of the continued fraction converges absolutely. For this type of convergence, which was suggested by a paper of Lane and Wall [Trans. Amer.

Math. Soc. 67, 368–380 (1949); these Rev. 11, 244], the author obtains two sets of necessary and sufficient conditions (Theorems 3.1, 3.2). Positive definiteness is defined for continued fractions (1) and sufficient conditions for absolute convergence of continued fractions equivalent to positive definite continued fractions are obtained. Proofs involve the determination of a sequence of closed nested circular regions which is suitably related to the sequence of linear fractional transformations $\{t_n(u)\}$ which generate (1), where $t_1(u)=b_0+a_1/(b_1-u)$, $t_{n+1}(u)=t_n[a_{n+1}/(b_{n+1}-u)]$, $n=1, 2, \cdots$. *W. T. Scott* (Evanston, Ill.).

Lane, Ralph E. A complete solution of the convergence problem for continued fractions. Proc. Amer. Math. Soc. 3, 914–920 (1952).

The author considers continued fractions of the form (1) $b_0+a_1/b_1-a_2/b_2-\cdots-a_n/b_n-\cdots$, where b_n, a_{n+1} are complex numbers, $a_{n+1}\neq 0$, $n=0, 1, 2, \cdots$ and proves the following theorem. A continued fraction (1′) has only finite approximants and converges if and only if there exists an equivalent continued fraction (1) and a sequence of numbers s_n, $n=1, 2, \cdots$, satisfying the following conditions: (i) $0<s_n<1$, $n=1, 2, \cdots$; (ii) $s_1|b_1|>|b_1-1|$, and (iii) $e_n\geq 1$, $n=1, 2, \cdots$, and $\sum_1^\infty (e_n-1)$ diverges, where

$$e_n=[s_{n+1}(1-s_{n+1}^2)^{-1}-|a_{n+1}-b_{n+1}+1(1-s_{n+1}^2)^{-1}|]/s_n|a_{n+1}|.$$

The proof depends on adroit use of the familiar "nest of circles" technique. *W. T. Scott* (Evanston, Ill.).

Mohr, Ernst. Elementarer Beweis für die Partialbruchzerlegung des Cotangens. Z. Angew. Math. Mech. 33, 247–248 (1953).

Fourier Series and Generalizations, Integral Transforms

Turán, P. On a trigonometrical sum. Ann. Soc. Polon. Math. 25 (1952), 155–161 (1953).

The author gives two general theorems (the first is a corollary of the second), each of which contains the positivity of the sums (1) $\sum_{\nu=1}^n \nu^{-1}\sin\nu t$ on $(0, \pi)$. (I) If $\sum_{\nu=1}^n b_\nu \sin(2\nu-1)x>0$ on $(0, \pi)$, then $\sum_{\nu=1}^n \nu^{-1}b_\nu \sin\nu x>0$ also. (II) If a_ν are real numbers such that

$$(2)\qquad f(x)=\sum_{\nu=0}^n a_\nu\cos\nu x\geq 0 \quad\text{and}\quad \sum_{\nu=1}^n a_\nu=0,$$

then (3) $\sum_{\nu=1}^n \nu^{-1}(a_0+a_1+\cdots+a_{\nu-1})\sin\nu x>0$ on $(0, \pi)$. Theorem II is proved by a complex-variable method similar to that used by the author for (1) [J. London Math. Soc. 13, 278–282 (1938)]. Other results are that for $0<x<\pi$ and $n\geq 2$,

$$\sum_{\nu=1}^n \nu^{-1}\sin\nu x>4\sin^2\tfrac{1}{2}x\left\{\tan\frac{\pi-x}{2}-\frac{\pi-x}{2}\right\};$$

that if $\sum a_\nu=0$ and $f(x)$ in (2) satisfies $|f(x)|\leq M$, then $g(x)$ in (3) satisfies $|g(x)|\leq M(\pi-x)/2$; and that if $f(\theta)\sim\sum b_\nu\sin\nu\theta$ is odd in $(0, 2\pi)$, convex in $(0, \pi)$, and $f(r, \theta)=\sum b_\nu r^\nu\sin\nu\theta$ is the corresponding harmonic function, then $\sum_{\nu=1}^n b_\nu r^\nu\sin\nu\theta\leq 2f(r, \theta)$ in the upper half of the unit circle. *R. P. Boas, Jr.* (Evanston, Ill.).

Arbault, Jean. Sur l'ensemble de convergence absolue d'une série trigonométrique. Bull. Soc. Math. France 80, 253–317 (1952).

A set E is called an N-set if there is a trigonometric series $\sum\rho_n\sin(nx+\varphi_n)$ absolutely convergent on E with $\sum\rho_n=\infty$ [J. Marcinkiewicz, Bull. Sém. Math. Univ. Wilno no. 1, 19–24 (1938)]. If there are two sequences (ρ_n) and (κ_n) of positive real numbers such that $\sum\rho_n|\sin\kappa_n x|$ converges on E and $\sum\rho_n=\infty$, then E is an N-set. Marcinkiewicz has proved that the sum of two N-sets is not an N-set in general. The author gives a simple proof of this theorem and proves the following theorems: (1) the linear manifold generated by an N-set with rational coefficients is also an N-set; (2) the sum of an N-set and an enumerable set is an N-set; (3) if E is a perfect set with finite Hausdorff measure defined by the function $\log_2 x^{-1}/\log x^{-1}$, then the sum of E with any N-set is an N-set; (4) if E is an N-set, then $\xi E=(\xi x; x\in E)$ is also an N-set for any ξ.

A set E is said to be an N_0-set if there is a sequence (κ_n) of integers such that $\sum|\sin\kappa_n\pi x|$ converges on E. The author proves properties of N_0-sets similar to N-sets, i.e., (5) the sum of an N_0-set and an enumerable set is an N_0-set; (6) if E is an N_0-set, then ξE is also; etc.

For a positive number p and a set E, if there is a series $\sum\rho_n|\sin n\pi x|^p$ convergent on E such that $\sum\rho_n=\infty$, the author says that E is an N^p-set. N_0^p-sets may be defined similarly. Then: (7) an N^p-set is an N-set; (8) the sum of an N_0^p-set and an enumerable set is an N_0^p-set; (9) if $0<p<q$, there is a set of N_0^q-type, but not of N_0^p-type. This is a generalization of a theorem due to R. Salem [Duke Math. J. 8, 317–334 (1941); these Rev. 2, 360].

Let $f(x)$ be an integral over the interval $(0, x)$ of a function positive and increasing in $x>0$ and vanishing at $x=0$. For a set E, if there is a series $\sum f(|\sin\kappa_n\pi x|)$ convergent on E, then E is said to be an N_0'-set. If $f(x)=x^p$, then an N_0'-set becomes an N_0^p-set. The author proves that: (10) an N_0'-set is not a group in general; (11) an N_0'-set is an N-set, but the converse does not hold, i.e., there is a perfect N-set not of type N_0'; (12) if f_1 and f_2 are functions satisfying the above conditions and $f_1<f_2$, then there is a set of type N_0', but not of type $N_0'^2$.

A sequence (κ_n) is said to be a null sequence with respect to a set E if $\sin\kappa_n\pi x\to 0$ $(n\to\infty)$ for x in E. Concerning this notion, it is proved that: (13) if (2^{n^2}) is a null sequence with respect to a set E, then any null sequence (λ_n) with respect to E is of the form $\lambda_n=\sum_{i=l_n}^{m_n}\alpha_i^{(n)}2^{i^2}$ where $m_n>l_n\to\infty$, $\sum_{i=l_n}^{m_n}|\alpha_i^{(n)}|=O(1)$; (14) if $\rho_n\to 0$, $\sum\rho_n=\infty$, then there are no null sequences with respect to the set $(x; \sum\rho_n|\sin 2^n\pi x|<\infty)$; (15) there are no null sequences with respect to the linear manifold generated by the set $(x; \sum|\sin 2^{n^2}\pi x|<\infty)$.

Further, the author proves the following theorems: (16) if $\sum\rho_n=\infty$, $\sum\rho_n\kappa_n/\kappa_{n+1}<\infty$, then $\sum\rho_n|\sin\kappa_n\pi x|$ converges on a perfect set; (17) a trigonometrical series converges uniformly on a perfect set P when it converges absolutely on P and P is covered by the sum of any portion, non-empty meet of P and an interval, with a finite number of its translations [cf. P. Malliavin, C. R. Acad. Sci. Paris 228, 1467–1469 (1949); these Rev. 10, 603]; (18) Cantor's ternary set is not the sum of two N-sets. *S. Izumi* (Tokyo).

Džvaršelišvili, A. G. Some properties of Fourier-Denjoy series. Soobščeniya Akad. Nauk Gruzin. SSR 13, 3–8 (1952). (Russian)

Let $f(x)$ be (D)-integrable on $(-\pi, \pi)$ and

$$(*)\qquad f(x)\sim\tfrac{1}{2}a_0+\sum(a_k\cos kx+b_k\sin kx).$$

Then: (i) The series $2\pi\sum b_k/k$ converges to the sum $\int_{-\pi}^{\pi}f(x)(\pi-x)dx$; (ii) the series (*) may be integrated term by term over any interval $(A, B)\subset(-\pi, \pi)$; (iii) if $f(x)$ is (D^*)-integrable and $g(x)$ is a periodic function with a

derivative $g'(x)$ of bounded variation on $(-\pi, \pi)$, then $\int_{-\pi}^{\pi} f(x)g(x)dx = \frac{1}{2}\pi a_0\alpha_0 + \pi\sum(a_n\alpha_n + b_n\beta_n)$ where α_n, β_n are the Fourier coefficients of $g(x)$. *M. Cotlar.*

Salem, Raphaël, et Zygmund, Antoni. Sur les séries trigonométriques dont les coefficients ont des signes aléatoires. C. R. Acad. Sci. Paris **236**, 571–573 (1953).

This paper is concerned with properties enjoyed for almost all values of t by series (S) of the form

$$\sum_{n=0}^{\infty} r_n\Phi_n(t)\cos(nx-\alpha_n)$$

where the Φ_n constitute the system of Rademacher functions. Under various restrictions on the coefficients r_n, analogs for S are stated of the law of the iterated logarithm and the central limit theorem as applied to series of independent random variables. Also, conditions insuring the continuity of S and bounds on its partial sums are found in terms of the r_n. Generalizations of some of the results in several directions are indicated. *G. Klein.*

Juncosa, M. L., and Young, D. M. A uniform approximation to Fourier coefficients. Proc. Amer. Math. Soc. **4**, 373–374 (1953).

Soit $f(x)$ réelle, continue pour $-1 \leqq x \leqq 1$ sauf en un nombre fini de points où elle est bornée. Soient

$$a_n = \frac{1}{2}\int_{-1}^{+1} f(x)e^{-inx\pi}dx,$$

$$b_n(p) = \frac{1}{2p}\sum_{k=-p+1}^{p} f\left(\frac{k}{p}\right)e^{-inx k/p} \quad \text{pour } |n| \leqq p,$$

$$b_n(p) = 0 \quad \text{pour } |n| > p.$$

Alors $b_n(p)$ tend uniformément vers a_n pour $p \to \infty$.

M. Zamansky (Paris).

***Cheng, M. T. Uniqueness of multiple trigonometric series.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 774–776. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

Cf. Ann. of Math. (2) **52**, 403–416 (1950); these Rev. **12**, 174.

Alexits, G. Sur les sommes de fonctions orthogonales. Ann. Soc. Polon. Math. **25** (1952), 183–187 (1953).

The $\phi_k(x)$ form a normal orthogonal system over $\langle a, b\rangle$, and the c_k are arbitrary real numbers. If $s_n(x) = \sum_0^n c_k\phi_k(x)$ and if the $\sigma_n^{(\alpha)}(x)$ are their nth Cesàro means of order α, then it is shown that

(i) $$s_n(x) = o(\log n \cdot (L_3(p, \epsilon)\sum_0^n c_k^2)^{1/2}),$$

(ii) $$\sigma_n^{(\alpha)}(x) = o((L_n(p, \epsilon)\sum_0^n c_k^2)^{1/2})$$

hold for almost all x in $\langle a, b\rangle$, provided that $\alpha \geqq 1$. Here $L_n(p, \epsilon) = \log n \cdot \log_2 n \cdots \log_p n \cdot (\log_{p+1} n)^{1+\epsilon}$, $\epsilon > 0$. The proof follows the usual lines; (ii) holds for all $\alpha \geqq 0$ when $n = 2^m$. [Cf. Kaczmarz and Steinhaus, Theorie der Orthogonalreihen, Warszawa-Lwów, 1935, p. 165; I. S. Gál, Ann. Inst. Fourier Grenoble **1**, 53–59 (1950); these Rev. **12**, 405.]

W. W. Rogosinski (Newcastle-on-Tyne).

Kovan'ko, A. S. On convergence of sequences of functions in the sense of Weyl's metric D_{W_ω}. Ukrain. Mat. Žurnal **3**, 465–476 (1951). (Russian)

For functions $f(x)$ and $\varphi(x)$ of L_ω defined for all real numbers, $D^T{}_{S_\omega}(f, \varphi)$ denotes the Stepanoff distance and $D_{W_\omega}(f, \varphi) = \lim_{T\to\infty} D^T{}_{S_\omega}(f, \varphi)$ is the Weyl distance. The space of functions with a finite Weyl norm is not complete. A sequence $\{f_n(x)\}$ is called uniformly D_{W_ω}-fundamental when there exists a function $T_0(\epsilon)$, $\epsilon > 0$, such that $\lim\sup_{n,m\to\infty} D^T{}_{S_\omega}(f_n, f_m) < \epsilon$ if $T > T_0(\epsilon)$. It is called uniformly D_{W_ω}-convergent to $f(x)$ if there exists a function $T_0(\epsilon)$, $\epsilon > 0$, such that $\lim\sup_{n\to\infty} D^T{}_{S_\omega}(f, f_n) < \epsilon$ if $T > T_0(\epsilon)$. The author proves that a sequence is uniformly D_{W_ω}-convergent if and only if it is uniformly D_{W_ω}-fundamental. He further proves by examples that a sequence can be uniformly D_{W_ω}-convergent without being $D^T{}_{S_\omega}$-convergent, and that it can be D_{W_ω}-convergent without being uniformly D_{W_ω}-convergent. A final example, proving that a sequence can be D_{W_ω}-fundamental without being D_{W_ω}-convergent, is very similar to one given by H. Bohr and E. Følner [Acta Math. **76**, 31–155 (1945); these Rev. **7**, 154].

H. Tornehave (Copenhagen).

Edmonds, Sheila M. The Parseval formulae for monotonic functions. IV. Proc. Cambridge Philos. Soc. **49**, 218–229 (1953).

In the earlier papers of this series [same Proc. **43**, 289–306 (1947); **46**, 231–248, 249–267 (1950); these Rev. **8**, 577; **11**, 592] Parseval formulae for pure sine and cosine Fourier transforms and series were established for functions in classes other than L_2. The following results are typical of the general intent of this paper to show that in certain directions these results cannot be improved. For example, if F_c and G_c are the respective cosine transforms of positive functions f and g in $L_1(0, \infty)$ which tend to zero monotonically in $(0, \infty)$, then

$$\int_0^\infty F_c(t)G_c(t)dt = \int_0^\infty f(x)g(x)dx$$

provided $fg \in L_1$. That this last restriction cannot be removed without compensation (say, by the convexity of f and g) as it can in the case of the sine transforms is shown by example. In the series case an instance is given which shows that the formula

$$\pi\left(\alpha_0 a_0 + 2\sum_{n=1}^{\infty}\alpha_n a_n\right) = 4\int_0^\pi f(x)g(x)dx$$

need not hold even if α_n, a_n decrease to zero, where $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx$, and $g(x) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}\alpha_n\cos nx$, unless the series on the left is convergent. *G. Klein.*

Saksena, K. M. Inversion formulae for a generalized Laplace integral. Proc. Nat. Inst. Sci. India **19**, 173–181 (1953).

The integral transformation

$$f(s) = \int_0^\infty (st)^{m-1/2}e^{-t/2}W_{k,m}(st)\phi(t)dt$$

is said here to have been introduced by Varma (unpublished). The author obtains three inversion formulas which correspond to inversion formulas for the Laplace transformation. [In a slightly different notation, the transformation was discussed by C. S. Meijer, Nederl. Akad. Wetensch., Proc. **44**, 727–737, 831–839 (1941); these Rev. **3**, 38, 109;

and its relation to the Laplace transformation by the reviewer, Univ. e Politecnico Torino. Rend. Sem. Mat. 10, 217–234 (1951); these Rev. 13, 937.] *A. Erdélyi.*

Mañas Díaz, J. On some theorems on the Laplace transform. I. Derivation and integration. An. Real Soc. Españ. Fis. Quim. Ser. A. Fis. 48, 99–108 (1952). (Spanish. English summary)

Mañas Díaz, Justo. On some theorems on the Laplace transform. II. Convolution and composition. An. Real Soc. Españ. Fis. Quim. Ser. A. Fis. 48, 201–208 (1952). (Spanish. English Summary)

Formulas are derived for the Laplace transforms of derivatives of functions having finite discontinuities. Impulse functions are also considered. Transforms of integrals of the functions are expressed so as to display the effect of discontinuities. The convolution of two functions one of which is a step function or an impulse function of any order is also formulated. The results are illustrated by solving two problems in electric circuits in which the impressed voltages are periodic functions of time whose first derivatives have finite discontinuities. The results display the effect of those discontinuities. *R. V. Churchill.*

Delerue, P. Sur le calcul symbolique à n variables et sur les fonctions hyperbesséliennes. Ann. Soc. Sci. Bruxelles. Sér. I. 67, 83–104 (1953).

This is an expository article on operational calculus in n variables and some of its applications. No references are given, and no essentially new results seem to be described. *A. Erdélyi.*

Polynomials, Polynomial Approximations

Sz.-Nagy, Gyula. Wertverteilung bei Polynomen mit lauter reellen Nullstellen und Koeffizienten. Acta Math. Acad. Sci. Hungar. 3 (1952), 269–274 (1953). (Russian summary)

For a polynomial $f(z)=(z-a_1)\cdots(z-a_n)$ with the a_j real and $a_1\geqq a_2\geqq\cdots\geqq a_p\geqq\cdots\geqq a_n$ the author proves that $\mathrm{Re}\,[(-1)^p f(z)]>0$ inside each deltoid region D_p, which is symmetric in the real axis and which is defined for $p=1, 2, \cdots, n-1$ as having as main diagonal the line segment (a_p, a_{p+1}) and having the angles π/p and $\pi/(n-p)$ at a_p and a_{p+1}, but is defined for $p=0, n$ as the sector having the magnitude π/n and vertex a_1 or a_n and not containing any a_j. The proof follows from a study of $\arg(a_j-z)$ for various j. Similarly proved is that, if $z_0=x_0+iy_0$ $(y_0>0)$ and $z_k=z_0+2y_0\tan(\pi/kn)$, then $f(z)/f(z_0)\neq$ a positive number on segment (z_0, z_1) and $\neq$ a real number on segment (z_0, z_2), and that $f(z)/f(z_0)\neq$ a negative number in the circle

$$|z-z_0|=|y_0|\sin(\pi/n),\quad y_0\neq 0.$$

M. Marden (Milwaukee, Wis.).

Obrechkoff, Nikola. Sur les racines des équations algébriques. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie II. 47, 67–83 (1952). (Bulgarian. French summary)

The author proves the following two theorems and makes a variety of applications. (I) If $f(x)=a_0+a_1x+\cdots+a_nx^n$ with real coefficients has p roots x_j in the sector

$$|\arg x|<\pi/(n+2-p),$$

and if V is the number of variations of sign in the sequence of coefficients, then $V=p+2t$ where t is a non-negative integer. (II) If $f(x)$ in (I) has p roots in the sector $|\arg x|<\pi/(m+n+2-p)$ and $n-p$ roots in

$$|\pi-\arg x|<\pi/(m+p+2),$$

and if $g(x)=b_0+\cdots+b_mx^m$ with real coefficients has all its roots in the sector

$$|\pi-\arg x|<\min\{\pi/(p+2), \pi/(n+2-p)\},$$

then the Hadamard product $a_0b_0+a_1b_1x+\cdots+a_kb_kx^k$, $k=\min\{m, n\}$, has only real roots. The author also obtains the largest interval about a root of a real equation, in which Newton's method will give a sequence converging to the root. *A. W. Goodman* (Lexington, Ky.).

Koenig, J. Frank. On the zeros of polynomials and the degree of stability of linear systems. J. Appl. Phys. 24, 476–482 (1953).

This paper indicates how, by translations and rotations, the determinant criteria of Hurwitz and Bilharz-Frank can be used to find whether or not all the zeros of a real polynomial lie in a sector $|\arg(-z)|\leqq\beta<\pi/2$, in an infinite strip or in certain other polygonal regions. *M. Marden.*

Rényi, A., and Turán, P. On the zeros of polynomials. Acta Math. Acad. Sci. Hungar. 3 (1952), 275–284 (1953). (Russian summary)

The authors develop two Graeffe Rules, of which the less general is the following: Let $f_0(x)=a_{00}+a_{10}x+\cdots+a_{n0}x^n$ with $a_{n0}=1$ have the zeros $z_1, z_2, \cdots, z_n$ with

$$|z_n|\leqq|z_{n-1}|\leqq\cdots\leqq|z_1|.$$

Introduce the Graeffe transforms

$$f_k(x)=(-1)^n f_{k-1}(\sqrt{x})f_{k-1}(-\sqrt{x})=a_{0k}+a_{1k}x+\cdots+a_{nk}x^n.$$

Let $M=[n^2\log(2n^2)]+2$, let S_j be the sum of the jth powers of the zeros of $f_k(x)$, and let $T_k''=(\max|s_j|^{1/j})^{2^{-k}}$ for $j=1, 2, \cdots, M$. Then

$$n^{-2^{-k}}\leqq|z_1|/T_k''\leqq(1-n^{-1})^{-2^{-k}}.$$

To prove both Rules, the authors establish the theorem that, if $f(t)=\sum_{j=1}^n b_jw^t$, where all $b_j>0$ and

$$1=|w_1|\geqq|w_2|\geqq\cdots\geqq|w_n|,$$

and if $0<\epsilon<\min(1, 2(b_2+\cdots+b_n)/b_1)$, $A=f(0)/(b_1\epsilon)$ and $M_1=[A\log 2A]+2$, then $\max|f(t)|\geqq b_1(1-\epsilon)$ for t an integer, $1\leqq t\leqq M_1$. They also extend this theorem from sums $f(t)$ to Fourier-Stieltjes integrals $\int_{-\infty}^{\infty}e^{itx}dF(x)$.

M. Marden (Milwaukee, Wis.).

***Turán, Pál. Sur l'algèbre fonctionnelle.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 267–290. Akadémiai Kiadó, Budapest, 1952. (Hungarian and French. Russian summary)

The author announces a few interesting results concerning polynomials arranged in terms of Hermite polynomials:

$$f(z)=\sum_{\nu=0}^{n}b_\nu H_\nu(z),\quad H_\nu(z)=(-1)^\nu e^{z^2}\left(\frac{d}{dz}\right)^\nu e^{-z^2},$$

some of them being analogous to certain classical theorems on polynomials arranged in terms of the powers z^ν. All zeros of $f(z)$ are in the strip $|\Im z|\leqq\frac{1}{2}(1+|b_n|^{-1}B)$ where

$B=\max|b_\nu|$, $0\leqq\nu\leqq n-1$, and also in the strip

$$|\Im z|\leqq\sum_{\nu=0}^{n-1}\left|\frac{b_\nu}{b_n}\right|^{\frac{1}{n-\nu}}.$$

Assuming that $f(-z)=f(z)$, n even, we have for the zeros,

$$|\Im z|\leqq 2^{1/2}(1+|b_n|^{-1}B')$$

where

$$B'=\max\{(\mu+1)(\mu+2)\cdots(\tfrac{1}{2}n)\}^{-1/2}|b_{2\mu}|,$$

$0\leqq\mu\leqq(\frac{1}{2}n-1)$. Further results deal with conditions on the coefficients b_ν involving the reality of all zeros. For instance, if all zeros of $f(z)$ are real, then the same holds for the zeros of $\sum_{\nu=0}^{n}b_\nu g(\nu)H_\nu(z)$ where $g(t)$ is a polynomial with only negative zeros. *G. Szegö* (Stanford, Calif.).

Brafman, Fred. A relation between ultraspherical and Jacobi polynomial sets. Canadian J. Math. **5**, 301–305 (1953).

The usual notation of the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x)=P_n(\alpha,\beta;x)$$

is employed. The author expresses $P_n(\alpha,\beta;x)$ in the following form

$$\sum_{k=0}^{n}c_kP_{n-k}(\tfrac{1}{2}(\alpha+\beta+k),\tfrac{1}{2}(\alpha+\beta+k);x).$$

The terms of this sum are ultraspherical polynomials. The coefficients c_k are given explicitely; they depend only on k (not on n). *G. Szegö* (Stanford, Calif.).

Makar, Ragy H. On algebraic simple monic sets of polynomials. Nederl. Akad. Wetensch. Proc. Ser. A. **56**= Indagationes Math. **15**, 250–257 (1953).

A set of polynomials $p_n(z)$ is simple and monic if $p_n(z)$ is of degree n and has leading coefficient 1, and algebraic if its coefficient matrix satisfies an algebraic equation. The author is interested in the question of what effect the property of being algebraic has on the known inequalities connecting the order (in the sense of J. M. Whittaker) of sets $\{p_n\}^\omega$ with the order ω of $\{p_n\}$. If the set is algebraic of degree 2, the order is preserved. If it is algebraic of degree 3, the order lies between $\frac{1}{2}\omega$ and 2ω, with equality attainable in all cases. This shows roughly that positive powers of algebraic sets behave better than those of general sets, while negative powers do not. *R. P. Boas, Jr.*

Ibragimov, I. I. On the best approximation of a function whose s-th derivative has a discontinuity of the first kind. Doklady Akad. Nauk SSSR (N.S.) **89**, 973–975 (1953). (Russian)

The author determines the asymptotic form (for large degree of the approximating polynomials) of the quantity described in the title, where s is positive but not necessarily integral. *R. P. Boas, Jr.* (Evanston, Ill.).

Ibragimov, I. I. On best approximation in the mean of a function whose s-th derivative is of bounded variation on the interval $[-1,1]$. Doklady Akad. Nauk SSSR (N.S.) **90**, 13–15 (1953). (Russian)

The author determines the asymptotic form of the quantity described in the title. He also considers the problem when the sth derivative has total variation at most 1, or when the $(s+1)$th derivative has the integral of its absolute value bounded by 1. *R. P. Boas, Jr.* (Evanston, Ill.).

***Nikol'skiĭ, S. M. Some questions of the approximation of differentiable functions.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27Août–2 Septembre 1950, pp. 113–124. Akadémiai Kiadó, Budapest, 1952. (Russian. Hungarian summary)

The author describes the results which he has published over the last five years on polynomial approximation of differentiable functions and its applications.

R. P. Boas, Jr. (Evanston, Ill.).

***Sz.-Nagy, Béla. Sur la convergence des séries de polynomes orthogonaux.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 249–258. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and French summaries)

This paper has appeared in a German version in Math. Nachr. **4**, 50–55 (1951); these Rev. **12**, 700.

***Alexits, György. The significance of the Lebesgue functions for the problem of convergence of expansions in orthogonal polynomials.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27Août–2 Septembre 1950, pp. 233–248. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian summary)

Let $\{p_n(x)\}$ be the orthonormal system of polynomials associated with the L-integrable weight $w(x)\geqq 0$, $a\leqq x\leqq b$. The series $\sum c_np_n(x)$ is almost everywhere convergent provided the following conditions are satisfied: (a) the series is almost everywhere $(C,1)$, (b) in any $a+\epsilon$, $b-\epsilon$, $\epsilon>0$, we have $|p_n(x)|\leqq M=M(\epsilon)$ uniformly, (c) $\int_a^{a+\epsilon}$ and $\int_{b-\epsilon}^{b}|p_n(x)|dx=O(\log n)$, (d) $\sum c_n^2\log n<\infty$. In a further part $\max|f(x)-s_n(x)|$, $-1+\epsilon\leqq x\leqq 1-\epsilon$, $\epsilon>0$, is estimated where $s_n(x)$ is the nth partial sum of the expansion of $f(x)$; it is assumed that $f^{(r)}(x)$ exists and satisfies a Lipschitz condition of exponent α, $0<\alpha\leqq 1$. The same difference is estimated if $\epsilon=0$, provided the total variation of $f^{(r)}(x)$ in -1, $+1$ is given. Finally expansions in terms of the Jacobi polynomials are studied in further detail.

G. Szegö (Stanford, Calif.).

Special Functions

Clemmow, P. C., and Senior, T. B. A. A note on a generalized Fresnel integral. Proc. Cambridge Philos. Soc. **49**, 570–572 (1953).

The function

$$G(a,b)=be^{ia^2}\int_a^\infty\frac{e^{-ib\lambda}}{\lambda^2+b^2}d\lambda$$

occurs in some recent work on the theory of radio propagation. The authors give the principal properties of this integral, show that $G(a,b)+G(b,a)$ can be expressed in terms of Fresnel's integral, give the first terms of series expansions when $|a|<|b|$ and $|a|>|b|$, and an asymptotic approximation for large $|a^2+b^2|$. *A. Erdélyi.*

Budden, K. G. A note on the Airy-integral function. Proc. Cambridge Philos. Soc. **49**, 574–577 (1953).

The author shows that one may arrive at an understanding of the behavior of Airy's integral by constructing amplitude-phase diagrams for points behind a lens on which a beam is incident. In particular, it turns out that reasonable approximations for points well beyond the caustic on the illuminated side lead to the asymptotic form of Airy's integral. *A. Erdélyi* (Pasadena, Calif.).

Agarwal, Ratan Prakash. A propos d'une note de M. Pierre Humbert. C. R. Acad. Sci. Paris **236**, 2031–2032 (1953).

Some results of Humbert's [same C. R. **236**, 1467–1468 (1953); these Rev. **14**, 872] are generalized to the function

$$E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{x^{m+(\beta-1)/\alpha}}{\Gamma(\alpha m+\beta)}.$$

A. Erdélyi (Pasadena, Calif.).

Bagchi, Hari das, and Mukherji, Bhola nath. Note on certain equations, connected with Bateman functions. Ann. Scuola Norm. Super. Pisa (3) **6** (1952), 269–280 (1953).

Bagchi, Hari das, and Mukherji, Bhola nath. Note on certain equations connected with Gegenbauer functions. Ann. Scuola Norm. Super. Pisa (3) **6** (1952), 281–290 (1953).

In some papers by Bagchi and others [cf. these Rev. **12**, 96, 178; **13**, 345, 554, 649, 842; **14**, 641] one finds comments on the relationships between differential equations and recurrence relations, etc., satisfied by the same function. In the present papers the corresponding comments are made in the case of Bateman's k function, and Gegenbauer polynomials, respectively. *A. Erdélyi* (Pasadena, Calif.).

Sengupta, H. M. On a certain definite integral. Math. Student **20** (1952), 122–123 (1953).

In this note the author evaluates $\int_{-1}^{1}[P_m(x)]^2dx$, where P_m is the Legendre polynomial, and gives some related results. *A. Erdélyi* (Pasadena, Calif.).

Friedman, Bernard, and Russek, Joy. Addition theorems for spherical waves. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-44**, i+19 pp. (1952).

Es handelt sich zunächst um die Entwicklung eines Produktes einer Besselschen Funktion erster Art, nullter Ordnung, einer abgeleiteten Legendreschen Funktion $n$ter Ordnung und einer Exponentialfunktion. Die entsprechenden Entwicklungsformeln werden abgeleitet und angegeben. Sodann wird die obige Besselsche Funktion erster Art, $n$ter Ordnung ersetzt durch Hankelsche Funktionen $n$ter Ordnung erster und zweiter Art. Einige der verwendeten Sätze werden in einem Anhang abgeleitet.

M. J. O. Strutt (Zürich).

Borkmann, K. Über einige Hilfsfunktionen mit zwei Eingängen, die bei Lösung der "Gleichung der eindimensionalen gedämpften Welle" auftreten. Z. Angew. Math. Mech. **33**, 216–217 (1953).

The author points out the usefulness of the functions

$$\tfrac{1}{2}[\varphi^{\alpha}\cosh(\alpha\varphi)\pm\varphi^{-\alpha}\cos(\alpha/\varphi)]$$

and

$$\tfrac{1}{2}[\varphi^{\alpha}\sinh(\alpha\varphi)\pm\varphi^{-\alpha}\sin(\alpha/\varphi)],$$

gives some properties of these functions, and states that they are being tabulated. *A. Erdélyi.*

Harmonic Functions, Potential Theory

Beckenbach, E. F., and Jackson, L. K. Subfunctions of several variables. Pacific J. Math. **3**, 291–313 (1953).

Let $\{\gamma\}$ be a family of closed contours γ bounding domains Γ whose closures $\bar{\Gamma}$ lie in a plane domain D, and let $\{\gamma\}$ include all circles having the same property and with radii less than a fixed $\rho>0$. Let $\{F\}$ be a family of real-valued functions satisfying: (1) for each $\gamma\in\{\gamma\}$ and function f continuous on γ there is a unique $F\in\{F\}$ equal to f on γ and continuous in $\bar{\Gamma}$; (2) if $M\geqq 0$, F_1, $F_2\in\{F\}$ and $F_1\leqq F_2+M$ on γ, then $F_1\leqq F_2+M$ in $\bar{\Gamma}$, and if the strict inequality holds at a point of γ it holds throughout Γ. A function g continuous in D is called a subfunction if $\gamma\in\{\gamma\}$, $F\in\{F\}$ and $g\leqq F$ on γ imply $g\leqq F$ in Γ. The authors obtain some properties of such generalized subharmonic functions and then impose two further restrictions. With this more restricted system the Dirichlet problem relative to $\{F\}$ is solved, i.e., sufficient conditions are found for the existence in $\{F\}$ of a unique solution of a boundary-value problem of Dirichlet type. The solution involves a criterion of Poincaré which is well known in connection with the classical Dirichlet problem. Applications to elliptic partial differential equations are promised. *F. F. Bonsall* (Newcastle-on-Tyne).

Górski, J. Sur un problème de F. Leja. Ann. Soc. Polon. Math. **25** (1952), 273–278 (1953).

$\{P_\nu\}$ is an extremal system (transfinite diameter theory) of n points on a set F in space and $u_n(P)=1/n\sum(1/PP_\nu)$. Does $u_n(P)$ converge as $n\to\infty$? Sets are known for which the answer is negative: here a positive answer for sets of capacity >0 is deduced from known results. The rest of the paper is unsound. *H. D. Ursell* (Leeds).

Diaz, J. B. Upper and lower bounds for quadratic functionals. Collectanea Math. **4**, no. 2, 3–49 (1951).

In dealing with various quantities of mathematical physics (for instance, with the electrostatic capacity, torsional rigidity, virtual mass, etc.) certain functionals appearing as "quadratic" integrals have to be extremized under suitable side conditions. In order to obtain bounds for these quantities, the author considers first certain analogous elementary problems for vectors. The following central theorem is quoted. Let c and w be arbitrary vectors, w_i orthonormal vectors, a_i real, $R\geqq 0$, and $|w-c|^2\leqq R^2$, $(w,w_i)=a_i$, $i=1,2,\cdots,n$. We write $(A,B,C\geqq 0)$

$$\sum a_i^2=A^2,\quad |c|^2-\sum(c,w_i)^2=B^2,\quad R^2-\sum[a_i-(c,w_i)]^2=C^2.$$

Then $|w|^2\leqq A^2+(B+C)^2$. If, in addition, $B>C$, we have $|w|^2\geqq A^2+(B-C)^2$. Finally, if $B\leqq C$, we have $|w|\geqq A$. This is applied to the estimation of Dirichlet's integral in the case of the following boundary problems: (a) Neumann's problem, (b) Dirichlet's problem, (c) the mixed problem (on the boundary $\alpha u+\beta\partial u/\partial n$ is given, where α, β are given continuous functions, $\alpha^2+\beta^2>0$, $\alpha\beta\geqq 0$). Finally two problems of the theory of elasticity are also treated from this point of view. *G. Szegő* (Stanford, Calif.).

Weinberger, H. F. Upper and lower bounds for torsional rigidity. J. Math. Physics **32**, 54–62 (1953).

Let R be a region bounded by an outer curve γ and by a finite number of inner curves γ_i. The torsional rigidity of S is defined by

$$S=\min\iint_R(\nabla f)^2dxdy,$$

where $\nabla^2 f=0$ in R and $\int_{\gamma_i}\partial f/\partial n\,ds=-2A_i$. The main tool of the author is the following simple inequality. Let the region R_{12} be the union of the disjoint regions R_1 and R_2. If the respective torsional rigidities are S_{12}, S_1, S_2, we have $S_{12}\geqq S_1+S_2$. From this remark (easily proved) various

upper and lower bounds are derived. Applications are given to domains bounded by certain simple curves, for example square, triangle, circle, hexagon with various kinds of holes.

G. Szegö (Stanford, Calif.).

Loewner, Charles. On generation of solutions of the biharmonic equation in the plane by conformal mappings. Pacific J. Math. **3**, 417–436 (1953).

Etude des fonctions biharmonique $u(x, y)$ (solutions de $\Delta\Delta u=0$) et de la fonction de Green $\Gamma(p, q)$ associée au problème des plaques encastrées (Γ est fonction biharmonique de p pour $p\neq q$, avec la singularité

$$\Gamma=|pq|^2\log|pq|+k(p),$$

$k(p)$ biharmonique régulière en q, et vérifie les conditions $\Gamma=\partial\Gamma/\partial n=0$ sur la frontière). Il est d'abord établi que les seules transformations dérivables $(x,y)\to(x',y')$, $u'=ux(x,y)$ qui laissent invariant l'ensemble des fonctions biharmoniques $u(x,y)$ est le groupe construit à partir de [$z'=(az+b)/(cz+d)$, $u'=ku|dz'/dz|$, a, b, c, d constantes, k constante réelle] et de $z\to\bar{z}$ ($z=x+iy$), résultat qui permet de poser correctement les problèmes où intervient le point à l'infini du plan en le ramenant à un point à distance finie. En utilisant la représentation de Goursat: $u(x,y)=\Re\{\bar{z}p(z)+q(z)\}$, $p(z)$ et $q(z)$ holomorphes, l'auteur caractérise $p(z)$ quand u est la fonction de Green $\Gamma(z,\infty)$ d'un domaine B ayant $z=\infty$ comme point intérieur. L'introduction de la représentation conforme $z=g(\zeta)$ de $|\zeta|>1$ sur B permet un calcul poussé de $f(\zeta)=p[g(\zeta)]$ quand $g(\zeta)$ est rationnel [cf. N. Moushelichvili, Application des intégrales analogues à celles de Cauchy . . . , Tiflis, 1922] et permet de donner une réponse à une question de J. Hadamard [Mém. Acad. Sci. Inst. France (2) **33**, no. 4 (1908)] concernant le signe de la fonction de Green: celle-ci peut effectivement changer de signe et prendre des valeurs négatives, ainsi que le montre l'exemple d'une famille de domaines non convexes étudiés par le procédé indiqué.

P. Lelong (Lille).

Szegö, G. Remark on the preceding paper of Charles Loewner. Pacific J. Math. **3**, 437–446 (1953).

En complément au mémoire précédent deux exemples de domaines plans pour lesquels la fonction de Green $\Gamma(z,\infty)=\Gamma(z)$ change de signe sont construits et étudiés en détail: le plan fendu suivant un arc du cercle unité d'amplitude $2\alpha<2\pi$ et un second domaine constitué par l'extérieur d'une courbe C_R, image de $|\zeta|=R$, $R>1$, dans la représentation $z=\phi(\zeta)$ de $|\zeta|>1$ sur le domaine à fente précédent avec conservation des points à l'infini.

P. Lelong (Lille).

Lüst, Reimar, Schlüter, Arnulf, und Trefftz, Eleonore. Verallgemeinerte Multipolfelder. J. Rational Mech. Anal. **2**, 485–517 (1953).

A vector field may (under certain conditions of continuity) be divided into a source-free field and a curl-free field. If the field is axially symmetric, the vectors of the curl-free field $\mathfrak{Q}$ all lie in the meridian planes. The source-free field may be divided into a part $\mathfrak{P}$ in the meridian planes and a part $\mathfrak{T}$ whose vectors are perpendicular to the axis. In constructing an ordinary multipole field we start from the potential of a point-charge $-1/r$. The force is then grad $(-1/r)=\mathfrak{r}/r^3$. The authors introduce the notations $\mathfrak{S}_0\{-1/r\}=-1/r$ and $\mathfrak{Q}_0\{-1/r\}=\mathfrak{r}/r^3$. For a bipole we find the potential field $\mathfrak{S}_1\{-1/r\}=(m\,\mathrm{grad}\,\mathfrak{S}_0\{-1/r\})=(m\mathfrak{r}/r^3)$, m being the unit vector in the direction of the axis, and the force $\mathfrak{Q}_1\{-1/r\}=\mathrm{grad}\,\mathfrak{S}_1\{-1/r\}$. Iterating these operations, we find for a 2^n-pole $\mathfrak{S}_n\{-1/r\}=(m\,\mathrm{grad}\,\mathfrak{S}_{n-1}\{-1/r\})$ and $\mathfrak{Q}_n\{-1/r\}=\mathrm{grad}\,\mathfrak{S}_n\{-1/r\}$. Instead of the scalar potential we may have a vector-potential. The vector-potential of a field $\mathfrak{P}_n$ in a meridian plane is a field $\mathfrak{T}_n$ so that $\mathfrak{P}_n\{-1/r\}=\mathrm{rot}\,\mathfrak{T}_n\{-1/r\}$ and in the field of a 2^n-pole we have of course $\mathfrak{P}_n(-1/r)=\mathfrak{Q}_n(-1/r)$. The generalisation that the authors have in view consists in replacing the function $-1/r$ from the ordinary potential by an arbitrary (sufficiently often derivable) function $g(r)$. Then

$$\mathfrak{S}_0\{g\}=g(r),\quad \mathfrak{S}_{n+1}\{g\}=(m\,\mathrm{grad}\,{}_n\mathfrak{S}\{g\}),$$
$$\mathfrak{Q}_n\{g\}=\mathrm{grad}\,\mathfrak{S}_n\{g\},\quad \mathfrak{T}_{n+1}\{g\}=[\mathfrak{Q}_n\{g\}\cdot m],$$
$$\mathfrak{P}_n\{g\}=\mathrm{rot}\,\mathfrak{T}_n\{g\}.$$

For small values of n the authors give explicit formulae for all these fields. Further, they prove a number of prescripts for calculations in such fields. In the last section they apply their results on an example from the theory of stellar magnetic fields. In an appendix to a paper of L. Biermann [Z. Naturforschung **5a**, 65–71 (1950)], A. Schlüter has given a model for the arising of magnetic fields on stars, where a very special solution of the plasma-equations may be found. Here there are given more general solutions.

H. Bremekamp (Delft).

Differential Equations

Tihonov, A. N. Systems of differential equations containing small parameters in the derivatives. Mat. Sbornik N.S. **31(73)**, 575–586 (1952). (Russian)

Consider a system (1) $dx/dt=f(x,z,t)$, $\mu dz/dt=F(x,z,t)$, where x, f are n-vectors, z, F m-vectors, and f, F satisfy suitable regularity assumptions. Assume $z=\varphi(x,t)$ is an isolated solution of $F(x,z,t)=0$ whose points are (asymptotically) stable equilibrium points of the adjoined system $dz/d\tau=F(x,z,t)$ (x, t being considered here as parameters, the point (x,t) belonging to a bounded open region D). Assume that the initial conditions (x^0,z^0,t^0) are such that the solution $z(\tau)$ of the adjoined system $dz/d\tau=F(x^0,z,t^0)$, $z(0)=z^0$, satisfies $\lim_{\tau\to\infty}z(\tau)=\varphi(x^0,t^0)$. Then the solution of (1) through the initial point (x^0,z^0,t^0) tends, as $\mu\to0$, to the solution of the degenerate system $dx/dt=f(x,\varphi(x,t),t)$ as long as the point (x,t) does not leave the region D. A similar result holds for systems containing several small parameters $\mu_1,\cdots,\mu_m$ tending to 0 in such a way that $\mu_{j+1}/\mu_j\to0$.

J. L. Massera (Montevideo).

Gradšteĭn, I. S. Application of A. M. Lyapunov's theory of stability to the theory of differential equations with small coefficients in the derivatives. Mat. Sbornik N.S. **32(74)**, 263–286 (1953). (Russian)

Let all quantities except t and η be finite-dimensional vectors. The author gives sufficient and highly complicated conditions in order that a solution of

$$\frac{dx}{dt}=f(x,y,t,\eta),\quad \eta\frac{dy}{dt}=h(x,y,t,\eta),$$

initiated at $x(0,\eta)$, $y(0,\eta)$ and converging to $x(0)$, $y(0)$ as $t\to0$, tend to the solution of the system obtained for $\eta=0$, initiated at $x(0)$, $y(0)$. Similar treatment for a system

$$\frac{dx}{dt}=f(x,y,z,t,\eta),\quad \eta\frac{dy}{dt}=g(x,y,z,t,\eta),$$
$$\eta^{1+\alpha}\frac{dz}{dt}=h(x,y,z,t,\eta),\quad \alpha>0.$$

Close connection is established with asymptotic stability à la Lyapunov.

[References: Gradšteĭn, Doklady Akad. Nauk SSSR (N.S.) **64**, 441–443; **65**, 789–792; **66**, 789–792 (1949); **82**, 5–8 (1952); Izvestiya Akad. Nauk. Ser. Mat. **13**, 253–280 (1949); these Rev. **10**, 536, 708, 709; **13**, 557; **10**, 709; Tihonov, A. N., Mat. Sbornik N.S. **27(69)**, 147–156 (1950); these Rev. **12**, 181; see also the preceding review.]

S. Lefschetz (Princeton, N. J.).

Gradšteĭn, I. S. On solutions on a time half-line of differential equations with small multipliers of the derivatives. Mat. Sbornik N.S. **32(74)**, 533–544 (1953). (Russian)

In several papers the author studied, through a finite time interval, the limiting performance of the solutions of a system in several vectors

$$\dot{X}=F(X, U, T, \eta), \quad \eta\dot{U}=H(X, U, T, \eta)$$

in relation to the solutions of

$$\dot{x}=F(X, U, T, 0), \quad 0=H(X, U, T, 0).$$

This is extended here to the time interval $[0-+\infty]$. [References: Gradšteĭn, Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 253–280 (1949); Doklady Akad. Nauk SSSR (N.S.) **82**, 5–8 (1952); these Rev. **10**, 709; **14**, 645; and the paper reviewed above.] *S. Lefschetz* (Princeton, N. J.).

Vasil'eva, A. B. On differential equations containing small parameters. Mat. Sbornik N.S. **31(73)**, 587–644 (1952). (Russian)

Consider the system (1) $\mu\dot{z}=F(z, y, t)$, $\dot{y}=f(z, y, t)$, where z is an n-vector, y an m-vector, $\mu\to 0$. Let $z=\varphi(y, t)$ be an isolated solution of $F=0$ which is a stable equilibrium point of the system (2) $dz/d\tau=F(z, y, t)$ for each fixed (y, t) in a region D; more precisely, assume that $F\cdot\varphi<0$ (scalar product) in a neighborhood of $z=\varphi$. Assume $(z^0, y^0, 0)$ is such that the solution $z(\tau)$ of (2) (for $y=y^0$, $t=0$) starting at z^0 tends to $\varphi(y^0, 0)$ when $\tau\to+\infty$. Let $\bar{y}$ be the solution of the system $\dot{y}=f(\varphi, y, t)$ satisfying $\bar{y}(0)=y^0$ and $\bar{z}=\varphi(\bar{y}, t)$ ($\bar{z}, \bar{y}$: solution of the "degenerate" system $0=F$, $\dot{y}=f$). It is known that if $z(t, \mu)$, $y(t, \mu)$ is the solution of (1) satisfying $z(0, \mu)=z^0$, $y(0, \mu)=y^0$, then, as $\mu\to 0$, we have $z(t, \mu)\to\bar{z}(t)$ for $0<t\leqq T$, $y(t, \mu)\to\bar{y}(t)$ for $0\leqq t\leqq T$. The present paper is devoted to the study of the convergence of the derivatives $\zeta=\partial z/\partial\mu$, $\eta=\partial y/\partial\mu$. Assume that F has continuous second derivatives and f continuous first derivatives and that the matrix A whose elements are the derivatives of F with respect to z is non-singular; let B be the matrix whose elements are the derivatives of F with respect to y and C, D the corresponding matrices whose elements are the derivatives of f. Then ζ, η satisfy the linear system

$$\mu\dot{\zeta}=A\zeta+B\eta-d\bar{z}/dt, \quad \dot{\eta}=C\zeta+D\eta, \quad \zeta(0)=\eta(0)=0,$$

and converge in $0<t\leqq T$ towards the solution of the degenerate system $0=\bar{A}\bar{\zeta}+\bar{B}\bar{\eta}-d\bar{z}/dt$, $d\bar{\eta}/dt=\bar{C}\bar{\zeta}+\bar{D}\bar{\eta}$ (where $\bar{A}$, $\cdots$ denote the values of A, $\cdots$ when z, y are replaced by $\bar{z}$, $\bar{y}$) satisfying the condition

$$\bar{\eta}(0)=\int_0^\infty\{f[z(\tau), y^0, 0]-f[\varphi(y^0, 0), y^0, 0]\}d\tau,$$

where $z(\tau)$ is the previously mentioned solution of (2). Hence η has, for small μ, a jump equal precisely to $\bar{\eta}(0)$; ζ is unbounded in the neighborhood of the origin. This theorem is generalized to the case of a system of the form $\dot{y}=f$, $\mu_\alpha\dot{z}_\alpha=F_\alpha$, $\alpha=1, \cdots, p$, where y is an m-vector, z_α an n_α-vector and the μ_α depend on a parameter μ in such a way that $\mu_\alpha/\mu_{\alpha+1}\to 0$ when $\mu\to 0$. *J. L. Massera.*

Volosov, V. M. On the theory of nonlinear differential equations of higher orders with a small parameter in the highest derivative. Mat. Sbornik N.S. **31(73)**, 645–674 (1952). (Russian)

Volosov, V. M. On solutions of some differential equations of the second order depending upon a parameter. Mat. Sbornik N.S. **31(73)**, 675–686 (1952). (Russian)

The results of a previous paper by the same author [Mat. Sbornik N.S. **30(72)**, 245–270 (1952); these Rev. **14**, 276] are generalized in the following way. In the first paper, equation (1) $\mu y^{(n)}+Q(x, y, y', \cdots, y^{(n-2)})=0$ is investigated; let $y^{(n-2)}=f(x, y, y', \cdots, y^{(n-3)})$ be a solution of the "degenerate" equation $Q=0$; assume that f has bounded continuous partials up to the third order and Q up to the second order in a region $x_0\leqq x\leqq\bar{x}, y, y', \cdots, y^{(n-2)}$ arbitrary; assume that $m|y^{(n-2)}-f|\leqq|Q|\leqq M|y^{(n-2)}-f|$, m, M positive, $Q>0$ (<0) when $y^{(n-2)}>f$ ($<f$). Then, if

$$y_0^{(n-2)}\neq f(x_0, y_0, \cdots, y_0^{(n-3)}),$$

the solution $y(x, \mu)$ of (1) through $(x_0, y_0, \cdots, y_0^{(n-1)})$ and its first $n-3$ derivatives converge uniformly, as $\mu\to 0$, towards a function $\varphi_0(x)$ and its derivatives, but $y^{(n-2)}(x, \mu)$ oscillates with a frequency of order $1/\mu^{1/2}$ between maxima and minima which, as $\mu\to 0$, tend to points on fixed curves C_1 and C_2; φ_0 and the equations of these curves can be explicitly calculated. If $y_0^{(n-2)}=f(x_0, y_0, \cdots, y_0^{(n-3)})$, the solution $y(x, \mu)$ and its first $n-2$ derivatives converge uniformly to the solution of the degenerate equation $y^{(n-2)}=f$ and its derivatives. A formula is also given for the evaluation of $\lim\int_a^b\psi(x, y, y', \cdots, y^{(n-2)})dx$, ψ being any continuously differentiable function. In the second paper, essentially the same results are proved for an equation

$$\mu[y''+\varphi(x, y, y')]+\Phi(x, y)=0$$

under the assumptions: $\Phi>0$ (<0) when $y>f(x)$ (<0) where f has a continuous second derivative; Φ has bounded continuous partials up to the second order, φ up to the first order; $m|y-f|\leqq|\Phi|\leqq M|y-f|$, m, M positive; φ/y' converges uniformly with respect to (x, y) to functions $\psi_1(x, y)$, $\psi_2(x, y)$ as $y'\to+\infty$, $y'\to-\infty$, and ψ_1, ψ_2 have bounded continuous partials of the first order. The last assumption is essential; an example for this is $\mu(y''+y'^2)+y=0$.

J. L. Massera (Montevideo).

Andreev, A. F. Solution of the problem of the center and the focus in one case. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 333–338 (1953). (Russian)

Lyapunov in a paper attached to the Russian edition of his fundamental mémoire, General problem of the stability of motion [Gostehizdat, Moscow-Leningrad, 1950; these Rev. **12**, 612], considered the nature of the critical point at the origin of the planar system

$$\begin{aligned}\dot{x}&=y+Ax^2+Bx^2y+Cxy^2+Dy^3,\\ \dot{y}&=Kx^2+Lx^2y+Mxy^2+Ny^3.\end{aligned}\tag{1}$$

He showed among other results that if

$$K<0, \quad 3A+L=0,\tag{2}$$

the point is a center or a focus. The author shows that a necessary and sufficient condition for a center are the conditions (2) and in addition

$$\begin{aligned}&2A(B+M)+K(C+3N)=0,\\ &(B+M)(2A^2-AKM-K^2N)=0.\end{aligned}\tag{3}$$

S. Lefschetz (Princeton, N. J.).

Krasovskiĭ, N. N. On stability of the solutions of a nonlinear system of three equations for arbitrary initial disturbances. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 339–350 (1953). (Russian)

Determination of a necessary and sufficient condition for asymptotic stability at the origin of a system

$$\dot{x}_i = f_i(x_1) + a_i x_2 + b_i x_3, \quad i = 1, 2, 3,$$

$|f_i(x)| < M|x|$ for x small. It is assumed that one of $a_1, b_1 \neq 0$ since otherwise the problem is elementary. *S. Lefschetz.*

Šimanov, S. N. On stability of solution of a nonlinear equation of the third order. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 369–372 (1953). (Russian)

Given the equation $\dddot{x} + f(x, \dot{x})\ddot{x} + b\dot{x} + cx = 0$, where b, c are constants, the author proves the theorem: If b, $c > 0$, $f(x, y) > c/b$, $yf_x' < 0$ for all x, y, then the origin is asymptotically stable for the equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f(x, y)z - bx - cy.$$

S. Lefschetz (Princeton, N. J.).

Hayashi, Chihiro. Subharmonic oscillations in nonlinear systems. Mem. Fac. Eng. Kyoto Univ. **14**, 206–224 (1952).

Hayashi, Chihiro. Subharmonic oscillations in nonlinear systems. J. Appl. Phys. **24**, 521–529 (1953).

(The content of these two papers is the same.) An investigation of subharmonic solutions of order 1/3 for $\ddot{v} + 2\delta\dot{v} + f(v) = B\cos 3t$, with $f(v) = c_1 v + c_3 v^3$ or $c_1 v + c_5 v^5$. Comparison with existing results [cf. M. E. Levenson, J. Appl. Phys. **20**, 1045–1051 (1949); these Rev. **11**, 439, and references given there] is lacking. *G. E. H. Reuter.*

Graffi, Dario. Sulle oscillazioni forzate nella meccanica non-lineare. Rivista Mat. Univ. Parma **3**, 317–326 (1952).

Assuming the existence of a solution $x(t)$ with period $T = 2\pi/\omega$ of the differential equation

$$\ddot{x} + \varphi(x)\dot{x} + \omega_0^2 x = C\cos(\omega t + \gamma)$$

the author shows that the inequalities

$$\frac{1}{T}\int_0^T x^2 dt < C^2/2(\omega^2 - \omega_0^2)^2, \quad \frac{1}{T}\int_0^T \dot{x}^2 dt < \omega^2 C^2/2(\omega^2 - \omega_0^2)^2,$$

are valid, provided $\omega > \omega_0$. From these inequalities a bound for $|x|$ is derived, which is used to prove that $x(t)$ is unstable, if $\varphi(x) < 0$ for $|x| < \pi C/(\omega^2 - \omega_0^2)$. The proofs make essential use of the Fourier series for $x(t)$. By similar methods inequalities are obtained for the Fourier coefficients of $x(t)$. These inequalities are used to appraise—in terms of the quantity $\max|\varphi(x)|\,(\omega^2 - \omega_0^2)^{1/2}$—the error committed if the friction term in the differential equation is neglected. Finally, the results, with the exception of the last mentioned error appraisal, are extended to the subharmonic periodic solutions, the existence of which is again assumed. *W. Wasow* (Los Angeles, Calif.).

Obi, Chike. Periodic solutions of non-linear differential equations of order $2n$. J. London Math. Soc. **28**, 163–171 (1953).

The author investigates the existence of periodic solutions of the system

$$\ddot{x}_r + \chi_r(x_r, \dot{x}_r^2) = k_{2r-1} f_{2r-1}(x_r, \dot{x}_r^2)\dot{x}_r + k_{2r} f_{2r}(x_r, \dot{x}_r^2)x_r + F_r(x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n, \epsilon, t) \quad (r = 1, \cdots, n).$$

Here χ_r, f_r, and F_r are analytic in all their arguments. $F_r \equiv 0$ when $\epsilon = 0$. Also $\epsilon = (\epsilon_1, \cdots, \epsilon_m)$ and $k = (k_1, \cdots, k_{2n})$ are small, F_r either is of period 2π in t or else independent of t. The degenerate system, with $\epsilon = k = 0$, is assumed to have a periodic, non-constant solution. Conditions for the existence of periodic solutions and expansion theorems for the non-degenerate system are given. *N. Levinson.*

Savinov, G. V. Auto-oscillations in essentially nonlinear quasi-conservative systems. Doklady Akad. Nauk SSSR (N.S.) **89**, 995–997 (1953). (Russian)

Certain quasi-conservative mechanical and electrical systems lead to the equation $\ddot{x} + g(x) = \mu f(x, \dot{x}, \mu)$, where μ is a small parameter. Assuming the existence of a periodic solution of this equation the author finds a first-order approximation to it by starting with the periodic solution $x_0(t)$ of the equation $\ddot{x} + g(x) = 0$ as a zero-order approximation. The functions $x_0(t)$, $x_0(t + \frac{1}{4}T)$, where T is the period of $x_0(t)$, are used for the construction of the first-order corrections, much as $\sin kt$, $\cos kt$ are used in the well known case $g(x) = k^2 x$. *M. Golomb* (Lafayette, Ind.).

Germay, R. H. Sur les intégrales régulières au sens de Fuchs de certains systèmes d'équations différentielles. Ann. Soc. Sci. Bruxelles. Sér. I. **67**, 69–76 (1953).

The author considers the system of differential equations

$$\frac{dy_j}{dx} + (x-a)^{-1}[f_{j1}(x)y_1 + \cdots + f_{jn}(x)y_n] = 0, \quad (j = 1, \cdots, n),$$

where the functions $f_{j1}(x), \cdots, f_{jn}(x)$, for each value of j, are holomorphic in a neighborhood of the point $x = a$, and are not all zero at that point. Solutions are obtained by an appropriate generalization of the classical method given by Fuchs for the case of a single equation of the nth order. The generalization involves considerable complications of the formalism, but no conceptual novelties. *L. A. MacColl.*

Magnus, Wilhelm. Infinite determinants in the theory of Mathieu's and Hill's equations. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **BR-1**, i+37 pp. (1953).

Der. Verf. geht von der Mathieuschen Differentialgleichung in der trigonometrischen Form aus und leitet die Hillsche Determinante für die Berechnung der charakteristischen Koeffizienten ab. Hierauf befasst er sich mit der besonderen Klasse von unendlichen Matrizen und Determinanten, wie sie hier vorliegen. Mit Hilfe allgemeiner Eigenschaften dieser Matrizen und Determinanten gelingt es, eine Reihe von Sätzen über die Lösungen der Mathieuschen Differentialgleichung abzuleiten. Für die Mathieuschen Determinanten wird eine Majorante abgeleitet. Die Werte der Lösungen der Mathieuschen Gleichung für die Werte $\pi/2$ und π der unabhängigen Veränderlichen werden durch unendliche Determinanten ausgedrückt. Die Lösungen der Mathieuschen Differentialgleichungen werden mit Hilfe Fourierscher Integrale angegeben. Aus dem Vorangehenden werden einige Verallgemeinerungen in Bezug auf Hillsche Differentialgleichungen diskutiert. Die Ergebnisse werden auf die Konstruktion wellendurchlässiger Medien angewandt. *M. J. O. Strutt* (Zürich).

Fogel, K.-G. On the solution in integral series of the wave equation with Yukawa potential. Soc. Sci. Fenn. Comment. Phys.-Math. **16**, no. 11, 7 pp. (1952).

The equation $\psi'' - (k^2 + l(l+1)/x^2 - be^{-x}/x)\psi = 0$ with $\psi(0) = \psi(\infty) = 0$ is studied where l is an integer. Series

methods are used and special attention is given to finding the eigenvalues for the case $k=0$. (Here the author states convergence is poor when $l\neq 0$.) For $l=0$ the method used was originally given by Jost [Helvetica Phys. Acta **20**, 256–266 (1947); these Rev. **9**, 555]. *N. Levinson.*

Glazman, I. M. On the character of the spectrum of one-dimensional singular boundary problems. Doklady Akad. Nauk SSSR (N.S.) **87**, 5–8 (1952). (Russian)

Let $S(A)$ be the spectrum of an operator A and let $D(A)$ be the discrete part and $C(A)$ the continuous part of the spectrum, $S=D+C$. Let l be a differential operator on $(0, \infty)$ and $\bar{L}$ a corresponding self-adjoint operator. The author proves a number of theorems. I. If $q(x)\to 0$ as $x\to\infty$, then $S(\bar{L})$, with $l=(-1)^n(d/dx)^{2n}+q(x)$, contains $\lambda>0$. If $\omega(q)=\limsup q(x)-\liminf q(x)$, for $x\to\infty$, is finite, then for all $\lambda>0$, $C(\bar{L})\cap[\lambda, \lambda+\omega]\neq 0$. II. Let

$$l[y]=\sum_0^n(-1)^{n-k}(p_k(x)y^{(n-k)})^{(n-k)}$$

and let $p_0(x)\to 1$, $p_k^{(r)}(x)\to 0$ for $k=0, 1, \cdots, n$; $r=0, 1, \cdots, k-1$. Then $S(\bar{L})\supset[0, \infty]$. III. Let II hold and

$$\lim p_k(x)=+0, \quad k=1, \cdots, n-1.$$

Then $(-\infty, 0)\cap C(\bar{L})=0$. Many further results are given. *N. Levinson* (Cambridge, Mass.).

Wolfson, Kenneth G. On the separation of spectra. Proc. Amer. Math. Soc. **4**, 408–409 (1953).

Let (1) $(p(t)x')'+(q(t)+\lambda)x=0$ be of limit-point type at $t=-\infty$ and $t=+\infty$. Then (1) determines a boundary-value problem on $L^2(-\infty, \infty)$, and together with $x(0)=0$ two boundary-value problems on $L^2(-\infty, 0)$ and $L^2(0, \infty)$. Let these three problems have spectra S, S_1 and S_2. It is proved that if λ' and λ'' belong to one of the sets S, S_1+S_2, then the closed interval λ', λ'' contains a point in the other set. *G. E. H. Reuter* (Manchester).

Phillips, Ralph S. Linear ordinary differential operators of the second order. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-42**, i+23 pp. (1952).

The Sturm-Liouville operator $Ly=-y''+q(x)y$ is studied where q is real and the boundary conditions are $y(0)=0$ and $\lim_{0<x\to\infty}(\alpha y+\beta y')=0$, where α and β are complex. In the limit-point case, L, quite naturally, turns out to be self-adjoint and independent of α and β. In the limit-circle case, L has a point spectrum which is in the upper half-plane if $\Im\alpha\beta^{-1}>0$ and in the lower half-plane if $\Im\alpha\beta^{-1}<0$. The operator L is also treated by a perturbation method when q is complex and $\Im q$ is bounded. *L. Gårding* (Lund).

Fage, M. K. On symmetry and symmetrizability of influence functions. Mat. Sbornik N.S. **32(74)**, 345–352 (1953). (Russian)

It is shown that Green's function $G(s, t)$ of a Sturm-Liouville equation on a bounded interval with general linear boundary conditions cannot be written in the form $H(s, t)r(t)$ with symmetric H and $r\neq 0$ unless G itself is symmetric. *L. Gårding* (Lund).

Karaseva, T. M. On the inverse Sturm-Liouville problem for a non-Hermitian operator. Mat. Sbornik N.S. **32(74)**, 477–484 (1953). (Russian)

Let $S_q(h, h_1)$ be the set of values of λ for which the equation (1) $-y''+q(x)y=\lambda y$ has a solution satisfying $y(0)=1$, $y'(0)=h$ and $y'(1)=h_1y(1)$. Continuing the work of Borg [Acta Math. **78**, 1–96 (1946); these Rev. **7**, 382] and Marčenko [Doklady Akad. Nauk SSSR (N.S.) **72**, 457–460 (1950); these Rev. **12**, 183] who considered real q, h and h_1, the author shows that two spectra $S_q(h, h_1)$ and $S_q(h, h_2)$ uniquely determine q, h and h_1 also in the complex case. The proof is based on the fact used by Marčenko that the solution $y=y(x, \lambda)$ of (1) for which $y(0)=1$ and $y'(0)=h$ can be written in the form $\cos\lambda^{1/2}x+\int_{-x}^{+x}g(x, \xi)\cos\lambda^{1/2}\xi d\xi$ with a continuous kernel g. *L. Gårding* (Lund).

***Bremekamp, H. Sur la théorie de Sturm-Liouville.** Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, pp. 38–56. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953.

An iterative technique in E. Picard's Traité d'analyse [v. III, 3rd ed., Gauthier-Villars, Paris, 1928] is generalized so as to yield a new proof for the existence of an infinite sequence of eigenvalues $\lambda=\lambda_n$, $n=1, 2, \cdots$, for boundary-value problems of the type

$$\frac{d}{dx}\left(K(x, \lambda)\frac{dy}{dx}\right)+G(x, \lambda)y=0, \quad y(a)=y(b)=0.$$

The following conditions are imposed. (1) $K(x, \lambda)$ and $G(x, \lambda)$ are continuous and positive in $a\leqq x\leqq b$, $\lambda\supset\lambda_0$, and $\partial K/\partial x$ is continuous there; (2) $K(x, \lambda)$ is independent of λ, or else a decreasing function of λ satisfying an inequality of the form $K(x, \lambda)<K(x, \lambda_0)/\psi(\lambda)$, where $\lim_{\lambda\to\infty}\psi(\lambda)=\infty$; (3) $G(x, \lambda)$ is an increasing function of λ such that $G(x, \lambda)>G(x, \lambda_0)\varphi(\lambda)$, where $\lim_{\lambda\to\infty}\varphi(\lambda)=\infty$. It is shown that the eigenfunctions corresponding to λ_n vanish $n-1$ times on $a<x<b$. An extension to more general linear boundary conditions is included. *W. Wasow.*

Sambasiva Rao, P. On eigenfunctions of the membrane problem. J. Indian Math. Soc. (N.S.) **17**, 1–20 (1953).

Let $\omega_1, \omega_2, \cdots$ be a complete orthonormal set of eigenfunctions of a vibrating membrane with fixed or free boundary covering a bounded open set D of k-dimensional space, having a smooth boundary. The series

$$\zeta(s)=\sum_{n=1}^{\infty}\omega_n(P)\omega_n(Q)\lambda_n^{-s-\frac{1}{2}\beta-\frac{1}{2}k}J_{\beta-\frac{1}{2}k}(R\lambda_n^{\frac{1}{2}}),$$

where λ_n is the eigenvalue corresponding to ω_n, is studied as a function of s. Only such points P and Q are considered for which the spheres with radius R and centers in P and Q are entirely contained in D. It is found that ζ is entire when $|P-Q|\neq R$. Otherwise it has simple poles on the real axis, depending on the values of β and k. When ω_n are the eigenfunctions of the Laplacean on the torus, the function ζ was studied by Bochner and Chandrasekharan [Quart. J. Math., Oxford Ser. **19**, 238–248 (1948); these Rev. **10**, 431]. *L. Gårding* (Lund).

Pini, Bruno. Sui punti singolari per i sistemi ai differenziali totali. Ann. Mat. Pura Appl. (4) **34**, 95–104 (1953).

The system studied is $dx_i=\alpha_i(x_1, x_2, x_3)du+\beta_i(x_1, x_2, x_3)dv$ $(i=1, 2, 3)$. It is assumed that the α's and β's are holomorphic, that $\det\partial\alpha_i/\partial x_j$ and $\det\partial\beta_i/\partial x_j$ are not zero, that $M=\|\alpha_i, \beta_i\|$ has in general rank 2, and that the system is passive. A point where rank $M<2$ is defined as singular of the first or second kind according as rank $M=1$ or 0. The behavior of the solution in the neighborhood of a singular point is studied by means of the Taylor series, the method

of treatment and results having analogies with the case of ordinary equations, as is pointed out. *J. M. Thomas.*

Karapandjitch, Djordje. Contribution aux méthodes d'intégration des systèmes des équations linéaires aux dérivées partielles. Bull. Soc. Math. Phys. Serbie **4**, no. 3–4, 29–37 (1952). (Serbo-Croatian. French summary)

The author asserts to have established a new extension of the method of variation of parameters to the solution of a linear system of partial differential equations in one unknown. In reality, the method consists in solving one or more equations of the system and substituting in the remaining equations. The method is claimed to be superior also for the reason that it gives integrals that other methods do not give. On examination these integrals turn out to be combinations of those obtained by the inferior methods.

M. Golomb (Lafayette, Ind.).

Baiada, Emilio. Considerazioni sull'esistenza della soluzione per un'equazione alle derivate parziali, con i dati iniziali, nel campo reale. Ann. Mat. Pura Appl. (4) **34**, 1–25 (1953).

Theorem: Let $f(x, q)$ be defined for $x \geqq c$ and $|q| < \infty$, be continuous in (x, q), and have $|f(x, q)| \leqq K$ and $|f(x, q') - f(x, q)| \leqq |q' - q|/2$. Let $\omega(y)$ $(|y| < \infty)$ have

$$|\omega'(y)| \leqq N \quad \text{and} \quad |\omega'(y+h) - \omega'(y-h)| \leqq |h| M(y),$$

with $M(y)$ summable in every finite interval. If $c < e < \infty$ and $-\infty < a < b < \infty$, then there exists for $c \leqq x \leqq e$ and $a \leqq y \leqq b$ a function $z(x, y)$ that is Lipschitzian in x and y together, has $z(c, y) = \omega(y)$ and, for each y in a set of measure $b - a$, has for almost all x derivatives z_x and z_y such that $z_x(x, y) = f(x, z_y(x, y))$. The author's approximation method [cf. Ann. Scuola Norm. Super. Pisa (2) **12**, 135–145 (1947); these Rev. **9**, 354] yields the result without use of the theory of ordinary differential equations. An example due to Pagni [Rend. Sem. Mat. Univ. Padova **20**, 470–474 (1951); these Rev. **13**, 845] shows that the solution is not unique.

F. A. Ficken (Knoxville, Tenn.).

Cinquini Cibrario, Maria, e Cinquini, Silvio. Ancora sopra una forma più ampia del problema di Cauchy per l'equazione $p = f(x, y, z, q)$. Ann. Scuola Norm. Super. Pisa (3) **6** (1952), 187–243 (1953).

The earlier paper [Ann. Mat. Pura Appl. (4) **32**, 121–155 (1951); these Rev. **13**, 845] treated the problem with data $z(0, y) = \phi(y)$ for the slab $0 \leqq x \leqq a$ in x, y, z, q-sphere. Here the underlying region is $0 \leqq x \leqq a$, $\alpha \leqq z \leqq \beta$, $\gamma \leqq q \leqq \delta$, and $g(x) \leqq y \leqq h(x)$ with $g(x) < h(x)$ for $0 \leqq x < a$; an existence and several uniqueness theorems relate to appropriate subregions. Methods (in particular, the use of characteristics) and conditions on f, ϕ, and the solution $z(x, y)$ resemble closely those of the earlier paper. In one theorem the domain of uniqueness depends on the data $\phi(y)$. *F. A. Ficken.*

Prodi, Giovanni. Soluzioni periodiche di equazioni alle derivate parziali di tipo parabolico e non lineari. Rivista Mat. Univ. Parma **3**, 265–290 (1952).

Let R denote the region $0 \leqq x \leqq l$, $-\infty < t, z, w < \infty$. Assume that in R the continuous function $f(x, t, z, w)$ is also periodic with respect to t, $f(x, t+T, y, z) = f(x, t, y, z)$. Let $\bar{f}(x, y, z)$ and $\underline{f}(x, y, z)$ denote respectively the maximum and minimum of $f(x, t, y, z)$ for $0 \leqq t \leqq T$. The function $\bar{f}$ $[\underline{f}]$ is assumed to be such that the differential equation $d^2y/dx^2 + \bar{f}(x, y, y') = 0$ $[d^2y/dx^2 + \underline{f}(x, y, y') = 0]$ has a solution $y = y_1(x)$ $[y = y_2(x)]$ such that $y_1(0)$ and $y_1(l)$ $[y_2(0)$ and $y_2(l)]$ are both positive [negative]. Also it is assumed that $y_1(x) < y_2(x)$. The principal result obtained in the present paper is that the nonlinear partial differential equation

$$u_t = u_{xx} + f(x, t, u, u_x), \quad 0 < x < l, \quad -\infty < t < \infty,$$

has at least one solution $u = u(x, t)$ which is periodic in t of period T if f satisfies the following additional condition. For every fixed number M if $|z| \leqq M$, then $\lim_{|w| \to \infty} f(x, t, z, w)/w^2 = 0$. The classical method of Leray and Schauder is used in proving the existence of a solution. For the solutions, using other methods, of problems similar to the above see the series of papers by Karimov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 403–406 (1940); **56**, 119–121 (1947); **58**, 969–972 (1947); these Rev. **2**, 204; **9**, 37, 287]. *F. G. Dressel* (Durham, N. C.).

Nirenberg, Louis. A strong maximum principle for parabolic equations. Comm. Pure Appl. Math. **6**, 167–177 (1953).

The strong maximum principle for harmonic functions states that the assumption of a maximum at an interior point implies constancy of the function. The present paper is concerned with similar principles for other classes of functions. Let $u = u(x_1, \cdots, x_n; t_1, \cdots, t_m)$ have continuous second derivatives in the $(n+m)$-dimensional domain T, and consider the differential operator

$$L(u) \equiv \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{\alpha,\beta=1}^{m} b_{\alpha\beta} u_{t_\alpha t_\beta} + \sum_{i=1}^{n} a_i u_{x_i} + \sum_{\alpha=1}^{m} b_\alpha u_{t_\alpha}.$$

The coefficients $a_{ij}, \cdots, b_\alpha$ are assumed continuous in T. $\sum a_{ij}\lambda_i\lambda_j$ and $\sum b_{\alpha\beta}\xi_\alpha\xi_\beta$ are respectively assumed to be positive definite and positive semidefinite forms. The component (in T) of the hyperplane $t_i = \text{constant}$ $(i = 1, \cdots, m)$ through point P will be denoted by $C(P)$. The following are typical of the two type of results obtained relative to the operator $L(u)$. (a) If u assumes in T its maximum at an interior point P_0 and $L(u) \geqq 0$ in T then $u \equiv u(P_0)$ on $C(P_0)$. (b) Assume $a \leqq 0$, u a solution of $L(u) + au = 0$ in T and that u has a positive maximum at a point P_0 in T, then $u \equiv u(P_0)$ on $C(P_0)$. Let $L'(u)$ denote the special case of $L(u)$ for $m = 1$, $b_{11} = 0$, $b_1 = -1$. By $S(P)$ denote the subset of T consisting only of points Q which can be joined to P by a simple curve in T along which the coordinate t_1 is non-decreasing. The following strong maximum principle is obtained for the parabolic operator $L'(u)$. If for some point P_0 of T the function u assumes on the set $S(P_0)$ its maximum at P_0, and if $L'(u) \geqq 0$ in $S(P_0)$, then $u \equiv u(P_0)$ on $S(P_0)$. Only one more of the many results along these lines will be given here. If $a \leqq 0$, u satisfies $L'(u) + au = 0$ in T, and if for some point P_0 of T, u achieves in the set $S(P_0)$ a positive maximum at point P_0, then $u \equiv u(P_0)$ in $S(P_0)$. *F. G. Dressel.*

Yosida, Kôsaku. On the Fokker-Planck equation and its integral. Sûgaku **3**, 129–136 (1951). (Japanese)

Yosida, Kôsaku. On the Fokker-Planck equation and its integral. II. Sûgaku **4**, 145–150 (1952). (Japanese)

The author considers the forward and backward diffusion equations in a Riemannian space and discusses their integrability conditions. In the first paper the author treats the problem as a problem in Banach spaces and obtains "operator-theoretical solutions", while in the second paper it is shown that these "operator-theoretical solutions" are actually solutions of partial differential equations having required properties and satisfying boundary conditions. The results of these papers have been published in four papers

(in English) of the author [J. Math. Soc. Japan 3, 69–73 (1951); Nagoya Math. J. 3, 1–4 (1951); Proc. Japan Acad. 27, 214–215 (1951); Proc. Amer. Math. Soc. 3, 864–873 (1952); these Rev. 13, 560, 656; 14, 560]. *S. Kakutani.*

Kamynin, L. I. On applicability of the method of finite differences to the solution of the equation of heat conduction. I. Uniqueness of solution of a system of finite-difference equations. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 163–180 (1953). (Russian)

The present paper contains detailed proofs of the results announced earlier by the author [Doklady Akad. Nauk SSSR (N.S.) 82, 13–16 (1952); these Rev. 14, 172]. These results were summarized in the previous review. In the particular case of the heat equation they center around the disparity in the uniqueness theorems for the initial value problem for the heat equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad t>0, \ -\infty<x<+\infty,$$

$$u(x,0)=\varphi(x), \quad -\infty<x<+\infty,$$

and the corresponding problem for the associated infinite system of differential-difference equations:

$$\frac{\partial u^{(h)}}{\partial t}(x,t)=\frac{1}{h^2}[u^{(h)}(x+h,t)-2u^{(h)}(x,t)+u^{(h)}(x-h,t)],$$

$$(x=\cdots,-2h,-h,0,h,2h,\cdots),$$

$$u^{(h)}(x,0)=\varphi(x).$$

J. B. Diaz (College Park, Md.).

Fulks, W. On the unique determination of solutions of the heat equation. Pacific J. Math. 3, 387–391 (1953).

Let $u=u(x,t)$ be a continuous, non-negative solution of the heat equation $u_{xx}=u_t$ on a rectangle $0<x<1$, $0<t<k$. It is known that the existence and identical vanishing of the normal boundary values $u(x,0+)$, $u(0+,x)$ and $u(1-,x)$ do not imply that $u\equiv 0$. However, the author proves that this is the case if $u(x,0+)=0$ for $0<x<1$ and if $u(x,t)$ tends to 0 when (x,t) tends to a boundary point $(0,s)$ or $(1,s)$, $0\leqq s<k$, along some parabolic arc $t-s=\alpha x^2$ or $t-s=\alpha(x-1)^2$, where α is positive (and can depend on the boundary point $(0,s)$ or $(1,s)$). The proof depends on an integral representation for u [Fulks, Thesis, Univ. of Minnesota, 1949; or P. Hartman and A. Wintner, Amer. J. Math. 72, 367–395 (1950); these Rev. 12, 104]. *P. Hartman.*

Manfredi, Bianca. Sopra un problema cilindrico non lineare di propagazione del calore. Rivista Mat. Univ. Parma 3, 383–396 (1952).

In three-dimensional Euclidean space let r denote the distance from the z axis. Let S and K denote respectively the regions $r>a>0$ and $0\leqq r<a$. The temperature $U=U(r,t)$ at time $t=0$ in S is zero, in K a gas is maintained at temperature 1. Following steps similar to those in a paper by W. R. Mann and F. Wolf [Quart. Appl. Math. 9, 163–184 (1951); these Rev. 13, 134; for additional results on this problem see J. H. Roberts and W. R. Mann, Pacific J. Math. 1, 431–445 (1951); these Rev. 13, 354] the author states U should satisfy the following four conditions: $U_t=U_{rr}+U_r/r$, for $r>a$, $t>0$; $U(r,0)=0$, $r\geqq a$; $U_r(a,t)=[U(a,t)-1]f(U(a,t))\equiv G(U(a,t))$, $t>0$; and $0<U<M<1$, $r\geqq a$, $t>0$. This is termed a nonlinear problem in the conduction of heat since the given function G is a function of U. It is assumed that $G(\xi)$ is a monotone increasing continuous function of ξ with $G(1)=0$. The Laplace transform is used to handle this boundary-value problem. The reviewer did not follow all the steps in the paper. *F. G. Dressel* (Durham, N. C.).

Zlateff, Ivan. Sur la propagation de la chaleur dans une barre se composant de deux parties de longueur variable dans des conditions linéaires aux limites. C. R. Acad. Bulgare Sci. 4 (1951), 5–8 (1953). (Russian, French summary)

The author gives a solution of this problem by reducing the problem to a system of two Volterra integral equations of the second kind. *C. G. Maple* (Ames, Iowa).

Delavault, Huguette. Sur un problème de la théorie de la chaleur, et sa solution au moyen des transformations de Hankel. C. R. Acad. Sci. Paris 236, 2484–2486 (1953).

The superposition of Laplace and Hankel integral transformations is used to solve the problem of the distribution of temperatures $f(t,r,z)$ in a slab $0\leqq z\leqq l$, $0\leqq r$, where r and z are cylindrical coordinates, when the initial temperature and the temperature on the boundary planes are prescribed functions. *R. V. Churchill* (Ann Arbor, Mich.).

Zatzkis, Henry. A certain problem in heat conduction. J. Appl. Phys. 24, 895–896 (1953).

A homogeneous spherical solid is surrounded by another homogeneous material of infinite extent. The entire solid is initially at temperature zero. Heat is generated uniformly throughout the sphere at a prescribed rate $Q(t)$. A formula is derived for the temperatures $U(r,t)$ in the composite solid. The derivation involves the continuity of the temperature at the interface but no condition on the behavior of the heat flux there. The boundary value problem is therefore incomplete and has no unique solution.

R. V. Churchill (Ann Arbor, Mich.).

***Pucci, Carlo. Maggiorazione degli integrali di equazioni differenziali lineari del seconde ordine.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 197–199. Casa Editrice Perrella, Roma, 1953.

Following some remarks on properties of solutions of partial differential equations, the author states two theorems which he published in an earlier paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 11, 334–339 (1951); these Rev. 13, 946]. *F. G. Dressel* (Durham, N. C.).

Višik, M. I. On systems of elliptic differential equations and their general boundary problems. Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 181–187 (1953). (Russian)

Author's summary of a thesis consisting of two articles: Mat. Sbornik N.S. 29(71), 615–676 (1951); Trudy Moskov. Mat. Obšč. 1, 187–246 (1952); these Rev. 14, 174, 473.

L. Gårding (Lund).

Garnir, H. G. Sur la solution élémentaire pour l'espace indéfini d'un opérateur elliptique décomposable du quatrième ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 1129–1141 (1952).

Soient A et B deux matrices carrées d'ordre n symétriques à coefficients réels. Appelons ∂ la matrice $(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, ${}^t\partial$ la matrice transposée. L'auteur considère l'opérateur différentiel elliptique d'ordre 4 décomposable

$$(\partial A\,{}^t\partial-k^2)(\partial B\,{}^t\partial-l^2).$$

Sa solution élémentaire s'exprime à partir de la solution élémentaire $G_k(r)$ de l'opérateur $(\Delta-k^2)^2$ $(k\neq 0)$,

$$G_k(r)=k^{\frac{1}{2}n-2}2^{1-\frac{1}{2}n}\pi^{-\frac{1}{2}n}r^{\frac{1}{2}n-2}K_{\frac{1}{2}n-2}(kr),$$

par la formule intégrale

$$\int_0^\infty [\text{dét. } D(\lambda)]^{-1/2}G_{(\lambda^2\lambda_1+t^2)^{1/2}}([xD^{-1}(\lambda)\,{}^tx]^{1/2})d\lambda,$$

où $D(\lambda)$ est la matrice $A+B\lambda$, x la matrice $(x_1, x_2, \cdots, x_n)$, et ${}^tx$ la matrice transposée. L'auteur retrouve ainsi des résultats de F. Bureau et M. Brillouin. *L. Schwartz.*

Gel'fand, I. M. On the spectrum of non-self-adjoint differential operators. Uspehi Matem. Nauk (N.S.) **7**, no. 6(52), 183–184 (1952). (Russian)

The author considers the spectrum of a non-self-adjoint differential operator on an unbounded domain. However, outside of some bounded domain the problem is self-adjoint. An example is given by (*) $-\Delta u+p_1u+p_2u=\lambda u$ where Δ is the Laplace operator, p_1 is a real function such that the operator $-\Delta+p_1$ is self-adjoint, and the function p_2 is complex-valued continuous and equal to zero outside of some compact domain. He proves that the spectrum of (*) consists of a real point set and at most a countable set of complex points. No cluster point of the spectrum is off the real axis. *N. Levinson* (Cambridge, Mass.).

Bureau, Florent. Les séries de fonctions fondamentales et les problèmes aux limites pour les équations aux dérivées partielles linéaires hyperboliques. Acta Math. **89**, 1–43 (1953).

This paper contains a detailed derivation of the results indicated in a previous note [C. R. Acad. Sci. Paris **234**, 791–792 (1952); these Rev. **13**, 750]. The author begins with a discussion of the solution $u(x, t)$ of the initial-boundary value problem for the equation $u_{tt}-L_x(u)=0$, first by a modified Riemann-method (which takes account of the boundary conditions) and then by expansion of u into eigenfunctions of L_x. He then proceeds to representations, which involve kernels, which can be represented by certain "fundamental series" of eigenfunctions. The most interesting formula for u of this type, which involves the function $K_s(x, y, t)$, has been quoted in the review mentioned above. The proof of that formula requires an analysis of the "zeta-function" for L_s

$$\zeta_s(x, y)=\sum_{k=1}^{\infty}\lambda_k^{-s}v_k(x)v_k(y)$$

as an analytic function of s. The study of ζ_s is reduced to that of the Green's function of the associated parabolic equation $\partial\theta/\partial t=L_x(\theta)$ [see Minakshisundaram, Canadian J. Math. **1**, 320–327 (1949); these Rev. **11**, 357]. It is proved that $\zeta_s(x, y)$ is an entire function of s for $x\neq y$, and has a simple pole at $s=1/2$ for $x=y$. In conclusion the author gives a proof that the function represented by $\sum_{k=1}^{\infty}\lambda_k^{-s}$ for sufficiently large $\mathrm{Re}(s)$ is regular analytic for $\mathrm{Re}(s)>-1/2$, except for a simple pole at $s=1/2$. *F. John.*

Babukov, A. G. On a boundary problem for the telegraph equation. Doklady Akad. Nauk SSSR (N.S.) **88**, 635–637 (1953). (Russian)

The problem is to find a solution of $u_{tt}+2\eta u_t=a^2u_{xx}$ for $0\leq x\leq l$, $t>0$, which admits the period T in t. In addition there is the condition $u(0, t)=0$, while for $x=l$ there are four t-intervals in which either u is given or else $u_x=0$. A trial expansion in terms of normal modes leads to an integral equation suitable, it is stated, to numerical treatment. The problem arises in the theory of pumps and of the vibration of bars. *F. V. Atkinson* (Ibadan).

Martin, M. H. The Monge-Ampère partial differential equation $rt-s^2+\lambda^2=0$. Pacific J. Math. **3**, 165–187 (1953).

The present paper is devoted to a study of the problem of Cauchy for the partial differential equation

(*) $$rt-s^2+\lambda^2=0, \quad \lambda=X(x)Y(y),$$

with a view to later applications to shock propagation. The result of the first section can be summed up in the theorem 1: The Monge-Ampère equation (*) has intermediate integrals only in the following cases:

	λ	Intermediate integrals
1.	0	$q=\varphi(p)$,
2.	1	$q-x=\varphi(p+y)$, $q+x=\psi(p-y)$,
3.	$Y(y)\neq 0$	$p+Y_1=\text{const.}$, $p-Y_1=\text{const.}$, $Y_1=\int Ydy$,
4.	y^{-2}	$px+qy-z-\frac{x}{y}=\varphi\left(p-\frac{1}{y}\right)$, $px+qy-z+\frac{x}{y}=\psi\left(p+\frac{1}{y}\right)$
5.	$\frac{x^{m-1}}{y^{m+1}}$ $(m\neq 0)$	$px+qy-z-\frac{1}{m}\left(\frac{x}{y}\right)^m=\text{const.}$, $px+qy-z+\frac{1}{m}\left(\frac{x}{y}\right)^m=\text{const.}$,
6.	$\frac{x^{-1}}{y}$	$px+qy-z-\lg\frac{x}{y}=\text{const.}$, $px+qy-z+\lg\frac{x}{y}=\text{const.}$,
7.	e^ze^v	$p-q+e^ze^v=\text{const.}$, $p-q-e^ze^v=\text{const.}$

and in those which arise from these under translations, reflections in the line $y=x$, and dilatations in the (x, y)-plane.

H. Lewy was the first to introduce characteristic variables α, β as independent variables to replace the characteristic differential system of a partial differential equation of second order by a system of partial differential equations of first order and to use them in the proof of existence and uniqueness of the solution to Cauchy's problem when intermediate integrals do not exist. Using those characteristic variables, the author proves the theorem 2: Whenever (*) has intermediate integrals, its integration can be reduced to the integration of linear partial differential equations of the first and second order, and quadratures. [This result can be expected from the general theory of Monge-Ampère differential equations; cf. E. Goursat, Leçons sur l'intégration des équations aux dérivées partielles du second ordre, t. 1, Hermann, Paris, 1896, p. 54.] In the problem of Cauchy a solution $z(x, y)$ of (*) is required such that along a prescribed curve (the carrier) $C: x=x(t)$, $y=y(t)$ in the (x, y)-plane the partial derivatives $p=z_x$, $q=z_y$ take the preassigned values (the Cauchy data) C_1: $p=p(t)$, $q=q(t)$. The author gets the theorem 3: Whenever the partial

differential equation (*) has intermediate integrals

$$\alpha=\alpha(x, y, z, p, q), \quad \beta=\beta(x, y, z, p, q),$$

the problem of Cauchy for this partial differential equation defines a carrier

$$\Gamma: \quad \alpha=\alpha(x(t), y(t), z(t), p(t), q(t))=a(t), \quad \beta=\beta(x(t), y(t), z(t), p(t), q(t))=\beta(t)$$

in the characteristic plane, and reduces to a problem of Cauchy with Cauchy data on this carrier for a linear partial differential equation of the second order. If Γ is a segment of one of the characteristic lines $\alpha=$const., $\beta=$const., or a point, the solution is not unique. *M. Pinl* (Dacca).

Forster, Herbert. Über eine Randwertaufgabe der Differentialgleichung

$$r^2\frac{\partial^2 f}{\partial r^2}+r\frac{\partial f}{\partial r}+q^2\frac{\partial^2 f}{\partial \varphi^2}=r^2k\frac{\partial f}{\partial t}.$$

Math. Z. **57**, 428–455 (1953).

The boundary value problem is: $r^2f_{rr}+rf_r+q^2f_{\phi\phi}=r^2kf_t$ for $r<R$, $\xi f+f_r=S(\phi, t)$ for $r=R$. Here r, ϕ are polar coordinates, t is time, q^2, k, ξ are (positive) constants, and $S(\phi, t)$ is a given function, periodic in t. The author first solves the problem for $k=0$ (the solution is called g) and then writes $f=g+\delta$ where δ will be small when k is small. The correction δ is expanded in a (double) series of Bessel functions (of non-integer order). The paper contains some results on the roots $x=z_{\mu\nu}$ of the equation

$$\alpha J_\nu(x)+xJ_\nu'(x)=0, \quad \nu\geqq 0, \quad \alpha\neq -\nu,$$

and inequalities for $J_\nu(z_{\mu\nu})$. *A. Erdélyi.*

Integral Equations

Krasnosel'skiĭ, M. A. Some properties of a root of a linear integral operator. Doklady Akad. Nauk SSSR (N.S.) **88**, 749–751 (1953). (Russian)

Let $K_r(s, t)$ be the rth iterate of the positive definite kernel $K(s, t)$, and assume $K_r(s, t)\in L^2\cap L^{p_0}$ $(p_0>2)$; then the integral operator A defined by K can be decomposed: $A=HH^*$, where H, H^* are adjoint linear operators, H completely continuous from L^2 to L^p, p being any number for which $2\leqq p<2rp_0/(2+rp_0-p_0)$. This result also holds with the obvious proper interpretation, when $p_0=\infty$.

M. Golomb (Lafayette, Ind.).

Bykov, Ya. V. On a class of linear integro-differential equations. Doklady Akad. Nauk SSSR (N.S.) **86**, 221–224 (1952). (Russian)

The author considers integro-differential equations of the form

$$(1) \qquad L[z(x)]=\varphi(x)+\lambda\int_a^b K(x, t)M[z(t)]dt,$$

where L is a linear differential operator of order n with continuous coefficients and leading coefficient 1, and M is a linear operator with appropriate (though complicated) continuity properties. By using the explicit solution of the differential equation $L[z(x)]=F(x)$ in terms of the Wronskian of a fundamental system $y_1, y_2, \cdots, y_n$ of solutions of the associated homogeneous equation, the author transforms (1) into an equation of the form

$$(2) \qquad z(x)=\sum_{k=1}^{n}c_ky_k(x)+\lambda\int_a^b H(x, t)M[z(t)]dt+f(x),$$

where the c_k are arbitrary constants. Applying the operator M to both sides of (2) and writing $M[z]=\psi$, he then reduces the problem of solving (1) to the solution of an ordinary integral equation

$$(3) \qquad \psi(x)=v(x)+\sum_{k=1}^{n}c_kv_k(x)+\lambda\int_a^b E(x, t)\psi(t)dt.$$

This enables him to show that the Cauchy initial value problem for (1) has a unique solution if and only if λ is not a characteristic value of the kernel $E(x, t)$. *F. Smithies.*

***Fenyő, István. A method of solution of nonhomogeneous linear integral equations.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 689–691. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian summary)

For a symmetrical kernel $K(x, y)$, such that the Fredholm theory applies and having known characteristic values λ_k $(|\lambda_0|\leqq|\lambda_1|\leqq\cdots)$, the author recommends for the solution of $\phi(x)=f(x)=\lambda\int_0^1K(x, y)\phi(y)dy$ the iterative process $\phi_0(x)=f(x)$,

$$(1-\lambda/\lambda_n)\phi_n(x)=f(x)+\lambda\int_0^1 K(x, y)\phi_{n-1}(y)dy-(\lambda/\lambda_n)\phi_{n-1}(x).$$

This is shown to converge for all $\lambda\neq\lambda_k$, with a rapidity related to the smoothness of the kernel; for a degenerate kernel it terminates. A non-symmetrical equation should first be symmetrised by a known device.

F. V. Atkinson (Ibadan).

Vaĭnberg, M. M. The existence of characteristic functions for nonlinear integral operators with nonpositive kernels and for the product of self-adjoint and potential operators. Mat. Sbornik N.S. **32(74)**, 665–680 (1953). (Russian)

The author gives a more detailed presentation of results and proofs published before [Mat. Sbornik N.S. **26(68)**, 365–394 (1950); **30(72)**, 3–10 (1952); Doklady Akad. Nauk SSSR (N.S.) **85**, 261–263 (1952); these Rev. **12**, 340; **13**, 658; **14**, 290]. *M. Golomb* (Lafayette, Ind.).

Functional Analysis

Zeller, Karl. *FK*-Räume in der Funktionentheorie. I. Math. Z. **58**, 288–305 (1953).

Diese Arbeit ist eine sehr leserliche Darstellung der Theorie der *FK*-Räume [vgl. Zeller, Math. Z. **53**, 463–487 (1951); diese Rev. **12**, 604] (das ist, der *F*-Räume im Sinne Bourbaki's, in denen Elemente komplexe Stellen $x=(x_1, \cdots, x_n, \cdots)$ sind und jede Abbildung $x\to x_n$ stetig ist) und enthält im wesentlichen keine neuen Resultate. Von Interesse ist dass die "Beweismethode der Produkträume [im wesentlichen von Banach und Schauder] diejenigen des "gleitenden Buckels" von Lebesgue [Ann. Fac. Sci. Univ. Toulouse (3) **1**, 25–117 (1909)] und der "halbstetigen Funktionen" von Saks [Banach und Steinhaus, Fund. Math. **9**, 50–61 (1927)] als ungefähr gleich weitführend an die Seite gestellt wird. Eine analoge Theorie, in der Stellen $(x_1, \cdots, x_n, \cdots)$ durch messbare Funktionen $x(t)$ ersetzt sind, steht noch aus. *G. G. Lorentz.*

Williamson, J. H. Linear transformations in arbitrary linear spaces. J. London Math. Soc. **28**, 203–210 (1953).

Let L be a linear space over the complex field K and L^* be its dual. Let $\mathfrak{S}(L)$ be the algebra of all endomorphisms of L. Then Q in $\mathfrak{S}(L)$ is called quasi-nilpotent if $|x^*(Q^n x)|^{1/n} \to 0$ for all x in L, x^* in L^*. The author gives necessary and sufficient conditions for T in $\mathfrak{S}(L)$ to possess a "classical spectral representation". By this it is meant (a) there exists a "spectral family" $E(\lambda)$ in $\mathfrak{S}(L)$ defined for each $\lambda=\mu+i\nu$ in K where (i) $x^*(E(\lambda)x)\to 0$ as $\mu\to-\infty$ and/or $\nu\to-\infty$, (ii) $x^*(E(\lambda)x)\to x^*(x)$ as $\mu, \nu\to\infty$ for all x in L, x^* in L^* and

(iii) $$E(\lambda_1)E(\lambda_2)=E(\inf(\mu_1,\mu_2)+i\inf(\nu_1,\nu_2)),$$

(b) $x^*(E(\lambda)x)$ defines a measure on the Borel sets of K, (c) there exists a quasi-nilpotent Q, $QT=TQ$ such that

$$x^*(Tx)=\int_K \lambda d(x^*(E(\lambda)x))+x^*(Qx)$$

for all x in L, x^* in L^*. One such necessary and sufficient condition is that to each x in L, x^* in L^* there corresponds a positive $k=k(x,x^*)$ where $|x^*(T^n x)|<k^n$ for all positive integers n. *B. Yood* (Eugene, Ore.).

Dieudonné, Jean. Sur un théorème de Smulian. Arch. Math. **3**, 436–440 (1952).

The Šmulian theorem [Mat. Sbornik N.S. **5**(47), 317–328 (1939); these Rev. **1**, 335] states that a closed convex subset K of a normed linear space E is weakly compact if and only if every decreasing sequence of closed convex subsets of K has a non-vacuous intersection. The author extends this criterion for weak compactness to locally convex spaces E which are quasi-complete in the sense that every bounded closed subset is a complete uniform space. An example is given showing that quasi-completeness is needed for the general theorem. A similar extension of a criterion due to Eberlein [Proc. Nat. Acad. Sci. U. S. A. **33**, 51–53 (1947); these Rev. **9**, 42; **10**, 855] was made by Grothendieck [Amer. J. Math. **74**, 168–186 (1952); these Rev. **13**, 857]. *C. E. Rickart* (New Haven, Conn.).

Tamura, Takayuki. Remarks on the convexity of connected sets. J. Gakugei Tokushima Univ. **3**, 24–27 (1953).

The principal theorem proved is that if M is a connected, locally convex subset of a separable real Banach space, then its interior relative to the smallest linear space spanning it is empty or convex. The paper follows an earlier note [same J. **1**, 25–30 (1950); these Rev. **13**, 468]. *M. M. Day.*

Cooke, Richard G. Generalizations of Banach-Hausdorff limits. Proc. Amer. Math. Soc. **4**, 410–417 (1953).

Generalizing a result of Eberlein [Proc. Amer. Math. Soc. **1**, 662–665 (1950); these Rev. **12**, 341] the author proves the existence of $(B\text{-}\mathfrak{A})$-limits, (and other similar limits) that is of those Banach limits on the Banach space m of bounded sequences which are invariant under all matrices A of a semi-group $\mathfrak{A}$ of positive matrices. The A's are assumed to be essentially commutable. In Eberlein's case, $\mathfrak{A}$ was generated by the translation matrix and by all positive regular Hausdorff matrices.

This result is essentially contained in a theorem proved by several authors [Kreĭn and Rutman, Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 3–95 (1948), Theorem 3.1; these Rev. **10**, 256; and an unpublished manuscript by the reviewer, Bull. Amer. Math. Soc. **58**, 183 (1952)]. In the form free of topology this theorem is as follows. Let S_0 be a sub-semigroup of a semi-group S (with operations of addition and multiplication with positive numbers), and let each difference $x-x'$ with $x, x' \in S_0$ defined in S belong to S_0. If $\mathfrak{A}$ is a commutative semi-group of linear operators A from S to S which leave S_0 invariant and if $l(x)$ is a positive linear functional on S_0, there exists a positive linear extension of $l(x)$ onto S which is A-invariant. In Cooke's theorem, S and S_0 are the factor spaces m/c_0 and c/c_0, where c and c_0 denote the sets of all convergent sequences and all sequences convergent to zero, respectively; $l(x)$ is the ordinary limit. The characterization of sequences for which all $(B\text{-}\mathfrak{A})$-limits (with a given $\mathfrak{A}$) coincide, obtainable in this way and given by the author is necessarily far from effective. It would be interesting to have conditions depending directly on the terms of the sequence, at least for some $\mathfrak{A}$.

G. G. Lorentz (Detroit, Mich.).

Alexiewicz, A. On the localization of values of vector valued functions. Ann. Soc. Polon. Math. **25** (1952), 288–297 (1953).

Let $\mathfrak{A}$ be a class of numerical functions (real or complex) defined over a set T and such that $\varphi \in \mathfrak{A}$ implies $\varphi+c \in \mathfrak{A}$ where c is any constant. Let A be a subset of T such that $\varphi(t)=0$, $t \in A$ implies $\varphi(t)=0$, $t \in T$. The principal theorem of the paper states that if $x(t)$ is a vector-valued function from T to a real or complex Banach space X, if L is a linear variety in X and if $\xi x \in A$ for each $\xi \in X^*$ then $x(t) \in L$ for each $t \in A$ implies $x(t) \in L$ for each $t \in T$. By specializing this theorem to the case in which T is a domain D in the complex plane and using various spaces X and appropriate linear varieties $L \subset X$ the author obtains vector-valued analogues of certain theorems in analytic function theory. In particular, the theorems of Vitali and Blaschke concerning convergence everywhere in D of sequences of functions converging on appropriate subsets of D are obtained and uniqueness theorems for analytic vector-valued functions whose values or limiting values at certain sets of points are specified are also proved as special cases of the principal theorem.

R. E. Fullerton (Madison, Wis.).

Alexiewicz, A., and Orlicz, W. On the differentials in Banach spaces. Ann. Soc. Polon. Math. **25** (1952), 95–99 (1953).

Let X and Y be two real Banach spaces and $y=F(x)$ a function from X to Y. It is shown that if $A \subset X$ is open and if F is continuous in A and if

$$\lim_{\tau\to 0}\frac{F(x+\tau h)-F(x)}{\tau}=\delta F(x,h)$$

exists for each $h \in X$, $x \in A$ then $\delta F(x,h)$ is linear and continuous in h for all $x \in A$ except for a set N of first category (i.e. is the continuous Gateaux differential of F in $A-N$). An example is given of a Lipschitzian function defined from an infinite-dimensional Banach space X to itself having at all points a Gateaux differential continuous in x and h jointly but not possessing a Fréchet differential at any point of X. *R. E. Fullerton* (Madison, Wis.).

Feller, William. On the generation of unbounded semigroups of bounded linear operators. Ann. of Math. (2) **58**, 166–174 (1953).

The differentiability and the (exp-) representability of the semi-group $\{T_t\}$ of bounded linear operators (E. Hille

and the reviewer) presupposes that T_t is strongly continuous not only for $t>0$ but also at $t=0$ with $T_0=I$, the identity. The author overcomes the assumption of the continuity at $t=0$. The result is important in view of the application to Cauchy's problem for partial differential equations. Let $\{T_t\}$, $t>0$, be a semi-group of bounded linear operators from a Banach space X to X such that: i) the range of $\{T_t\}$ is dense in X; ii) $\lim_{h\downarrow 0} T_{t+h}x=T_tx$ for $x\in X$; iii) $\|T_t\|$ is bounded in t. The author defines the continuity set Σ of $\{T_t\}$ as the set of those x for which $\lim_{t\downarrow 0} T_tx=x$. The infinitesimal generator Ω of $\{T_t\}$ is defined by

$$y=\Omega x=\lim_{h\downarrow 0} h^{-1}(T_hx-x), \quad x\in\Sigma,$$

if the latter limit exists. It is proved that Σ is dense in X and that, for $\lambda>0$ and $x\in\Sigma$, the resolvent equation $\lambda z-\Omega z=x$ has a unique solution $z=J_\lambda x$ satisfying

$$N(x)=\sup_{n\geq 1}\sup_{\lambda>0}\|\lambda^nJ_\lambda^nx\|<\infty.$$

Moreover, the set Σ remetrized by $N(x)$ is a Banach space such that $N(\lambda J_\lambda)\leqq 1$ and $\lim_{\lambda\to\infty} N(\lambda J_\lambda x-x)=0$. Conversely, let there be given an operator Ω defined from a set Σ dense in X to Σ such that its resolvent J_λ satisfies the above conditions. Then it is proved that there exists a semi-group $\{T_t\}$ of bounded linear operators from X to X which is strongly continuous for $t>0$ with the infinitesimal generator Ω. It is defined by $T_tx=\lim_{\lambda\to\infty}\exp(\lambda t(\lambda J_\lambda-I))x$.

K. Yosida (Osaka).

Feller, W. On positivity preserving semigroups of transformations on $C[r_1, r_2]$. Ann. Soc. Polon. Math. **25** (1952), 85–94 (1953).

The purpose of this paper is to give a new derivation of the backward equation of diffusion theory under weaker conditions than were formerly assumed [cf. Kolmogoroff, Math. Ann. **104**, 415–458 (1931); Feller, ibid. **113**, 113–160 (1936)]. The author considers semi-groups of transformations $\{T_t\}$ defined for $t>0$ on the Banach space $C[r_1, r_2]$ of real-valued functions continuous in $[r_1, r_2]$ and possessing the usual norm $\|x\|=\sup|x(s)|$. It is assumed that $\{T_t\}$ is strongly continuous for $t>0$ and that the operators are positivity-preserving contraction transformations. It is further assumed that the infinitesimal generator A is of a local character in the sense that for each $\xi\in[r_1, r_2]$ and any $x(s)\in C[r_1, r_2]$ which vanishes in some interval $|s-\xi|<\delta$ one has $h^{-1}T_h[x(s)]|_{s=\xi}\to 0$ as $h\to 0$. Finally, it is assumed for each $x(s)$ in the domain D of A that the derivatives $x'(s)$ and $x''(s)$ exist. The principal result states that at each regular point, the infinitesimal generator A coincides with the differential operator $\Lambda=a(s)d^2/ds^2+b(s)d/ds+c(s)$ where $a(s)\geqq 0$ and $c(s)\geqq 0$. Here a point $\xi\in[r_1, r_2]$ is said to be regular if given real numbers p_0, p_1, p_2 there exists an $x(s)\in D$ such that $x(\xi)=p_0$, $x'(\xi)=p_1$, and $x''(\xi)=p_2$.

R. S. Phillips (New Haven, Conn.).

Citlanadze, È. S. Existence theorems for minimax points in Banach spaces and their applications. Trudy Moskov. Mat. Obšč. **2**, 235–274 (1953). (Russian)

The first section of this paper discusses properties of Fréchet differentials of functionals and operators on a Banach space E: the differential of $f(x)$ is denoted by $df(x; h)$ and is equal to (L_fx, h), where L_f is an operator from E to its conjugate, if f is a functional. The major part of the paper is concerned with conditional extrema for functionals.

If $\{\varphi_i(x)\}$ is a finite set of functionals with first and second order Fréchet differentials, a point x on the surface R given by $\varphi_i(x)=a_i$ is called an ordinary point if $d\varphi_i(x; h)$ are independent as functionals on h, and is called a critical point for a functional $F(x)$ if $dF(x; h)$ is linearly dependent on these functionals. A class of compact subsets of R, say $[V]$, is called a compact homotopic class if it can be derived from one member by continuous deformations. Write $c(V)=\max\{F(x): x\in V\}$ and $c=\min\{c(V): V\in[V]\}$. It is shown that if there is a $V\in[V]$ with $c(V)=c$, then V contains a critical point of F for the value c.

Let $F(x)$ and $\varphi(x)$ be differentiable functionals satisfying a Lipschitz condition of order 1, let $dF(x; h)=(L_Fx; h)$ $d\varphi(x; h)=(L_\varphi x; h)$; a point x for the set U on which $\varphi(x)=a$ is called a proper element of L_F if $L_Fx=\lambda L_\varphi x$; if $\varphi(x)=\|x\|$ and E is Hilbert space this corresponds to the ordinary definition of proper elements. It is shown that for any $\alpha>0$ there is in any compact homotopic class a class containing an x such that $\|(L_\varphi x, x)L_Fx-(L_Fx, x)L_\varphi x\|<\alpha|(L_\varphi x, x)|$. If $(L_\varphi x, x)$ is bounded below by a positive number on U, if L_φ has a continuous inverse on U and L_F is a compact operator, then it is shown that in any compact homotopic class $[V]$ on U either (i) there is a proper value of L_F such that $L_Fx=\lambda L_\varphi x$ with $\lambda\neq 0$, $F(x)=c$, or (ii) there is a sequence $\{x_n\}\subset U$ such that $L_Fx_n\to 0$, $F(x_n)\to c$. Further conditions on the space and L_F eliminate the second possibility.

The theory of the Lyusternik-Šnirelman category is extended to infinite-dimensional spaces, in particular to the infinite-dimensional projective space S^* obtained by identifying diametrically opposite points in S, the unit sphere in E: from a theorem of Šnirelman it is deduced that S^* contains sets of all categories.

Let E be reflexive, have a basis, and let $\|x\|=\varphi(x)$ be differentiable; let $d\varphi(x; h)=(Nx; h)$ and let Nx satisfy a Lipschitz condition of order 1 and have a continuous inverse on S: let F be positive and even and L_F compact; let $F(y)=0$ if and only if $y=0$. F defines a function on S^*. Let $[V^*]_k$ be the class of closed compact sets of S^* having category $\geqq k$ in S^*, and let $c_k=\sup_{[V^*]_k}\min_{V^*}F(x)$. Then it is shown that the c_k decrease to 0. If $c_i=c_{i+p}$ the intersection of the sets in which $F(x)=c$ and $L_Fx=(L_Fx, x)Nx$ has category at least $p+1$.

The nonlinear integral operator

$$L_F\varphi=\sum_{n=1}^{\infty}\int_0^1\cdots\int_0^1 K_n(s, t_1, \cdots, t_n)\varphi(t_1)\cdots\varphi(t_n)dt_1\cdots dt_n,$$

where

$$\varphi\in L^2(I)=L(0, 1), \quad K_n(s)\in L^2(I^n),$$
$$\sum_n \max_s\|K_n(s)\|_{L^2(I^n)}<\infty,$$

was studied by Liechtenstein. By taking

$$F\varphi=\sum_{n=2}^{\infty}\frac{1}{n}\int_0^1\cdots\int_0^1 K_{n-1}(t_1, \cdots, t_n)\varphi(t_1)\cdots\varphi(t_n)dt_1\cdots dt_n,$$

it is brought within the scope of the present theory, and the existence of at least one, and under certain conditions of an infinite number, of proper values is proved. Applications to other nonlinear equations are also given.

J. L. B. Cooper (Cardiff).

Stein, Marvin L. Sufficient conditions for the convergence of Newton's method in complex Banach spaces. Proc. Amer. Math. Soc. **3**, 858–863 (1952).

Let $T(y)$ be a transformation defined on a sphere S: $\|y\|<\rho$ in a complex Banach space Y with values in an-

other such space Z. The author gives conditions under which the equation $T(y)=0$ may be solved by an analogue of Newton's method of successive approximations: $y_{i+1}=y_i-\delta T^{-1}(y_i; T(y_i))$ where $\delta T^{-1}(y; h)$ denotes the inverse with respect to h of the first variation $\delta T(y; h)$. Suppose that: (i) $T(y)$ is G-differentiable; (ii) there exists y_0 in S, $a>0$ and a sufficiently small $M>0$ (this is made precise) such that $\|T(y)\|\leqq M$ if $\|y-y_0\|\leqq a$; (iii) $T(y_0; h)$ is a 1-1 mapping of Y onto Z. Then the sequence given by Newton's method will converge to a solution $\bar{y}$ of $T(y)=0$ and is the unique solution of that equation in a neighborhood of $\bar{y}$. This work is related to that of Kantorovič [see, e.g., Trudy Mat. Inst. Steklov. **28**, 104–144 (1949); these Rev. **12**, 419] and Mysovskih [see, e.g., ibid. **28**, 145–147 (1949); these Rev. **12**, 419]. *B. Yood* (Eugene, Ore.).

Kračkovskiĭ, S. N. Canonical representation of null elements of a linear operator in its Fredholm region. Doklady Akad. Nauk SSSR (N.S.) **88**, 201–204 (1953). (Russian)

Let A be a linear bounded transformation of a complex Banach space into itself. Let $T_\lambda=I-\lambda A$, I the identity, and let Φ_A be the Fredholm region for A (see the review cited below for definitions). In an earlier paper of Gol'dman and Kračkovskiĭ [same Doklady (N.S.) **86**, 15–17 (1952); these Rev. **14**, 478] it was shown that the components of Φ_A may be divided into two mutually exclusive classes; each component in the second of these classes contains only λ's which are eigenvalues and for which the dimension of the set of null elements $N(\lambda)$ is infinite. For such a λ the author takes a basis for $N(\lambda)$ as follows:

$$x_k^{(1)}, x_k^{(2)}, \cdots \quad (k=1, 2, \cdots, s),$$
$$x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(r_k)} \quad (k=s+1, \cdots, s+p),$$

where

$$T_\lambda x_k^{(1)}=0, \quad T_\lambda x_k^{(2)}=-x_k^{(1)}, \cdots \quad (k=1, 2, \cdots, s),$$
$$T_\lambda x_k^{(1)}=0, \quad T_\lambda x_k^{(2)}=-x_k^{(1)}, \cdots,$$
$$T_\lambda x_k^{(r_k)}=-x_k^{(r_k-1)} \quad (k=s+1, \cdots, s+p).$$

Here p may be zero, but s cannot be zero. The author proves the following theorem: The number s is the same for all λ's from the same component in the second class. As the author points out, essentially the same result was also found by Gohberg [ibid. **78**, 629–632 (1951), p. 629; these Rev. **13**, 46]. *J. V. Wehausen* (Providence, R. I.).

Gol'dman [Goldmanis], M. A. Approximation of completely continuous linear operators by finite-dimensional ones and the Riesz-Schauder theory. Latvijas PSR Zinātņu Akad. Vēstis **1951**, no. 7 (48), 1151–1160 (1951). (Russian. Latvian summary)

The author proves the Fredholm theorems for completely continuous operators by first proving them for finite-dimensional operators and then using a limiting process.

J. V. Wehausen (Providence, R. I.).

Krasnosel'skiĭ, M. A. New existence theorems for solutions of nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) **88**, 949–952 (1953). (Russian)

The following theorem is stated. Let Φ be a function defined in the Hilbert space H and such that $\lim_{\|x\|\to\infty}\Phi(x)=\infty$. Let $\Gamma=\Gamma_1+\Gamma_2$ be the gradient of Φ, where Γ_1 is linear, bounded with a bounded inverse, and Γ_2 is completely continuous. Then $\Phi(x)$ attains its minimum at a point $x_0\in H$, and $\Gamma x_0=0$. This variational principle is used to prove the existence of a solution $\varphi\in L^p$ for Hammerstein's equation $\varphi(x)=\int_G K(x, y)f[y, \varphi(y)]dy$ where the kernel is symmetric and has a finite number of negative eigenvalues and $f(y, u)$ satisfies suitable conditions. *M. Golomb.*

Bade, William G. An operational calculus for operators with spectrum in a strip. Pacific J. Math. **3**, 257–290 (1953).

The author develops an operational calculus for closed linear operators T whose domain and range lie in a Banach space X, and whose spectrum is confined to a vertical strip $|\sigma|\leqq\gamma$ in the plane of the complex variable $\lambda=\sigma+i\tau$. He assumes also that the resolvent $R_\lambda(T)=(\lambda I-T)^{-1}$ satisfies

$$\|R_\lambda(T)\|\leqq M(t) \quad (|\sigma|\geqq t>\gamma).$$

Let $D(T)$ be the domain of T, and write

$$D_n(T)=\{x: x, Tx, \cdots, T^{n-1}x\in D(T)\} \quad (n\geqq 1).$$

Denote by $\mathfrak{L}(0, \gamma)$ the class of functions $f(\lambda)$ which are analytic in some strip $|\sigma|<r$, where $r>\gamma$, and such that $f(\sigma+i\tau)\to 0$ $(|\tau|\to\infty)$ uniformly in any strip $|\sigma|\leqq r_1<r$, and $\int_{-\infty}^{\infty}|f(\sigma+i\tau)|d\tau<\infty$ $(|\sigma|<r)$. Finally, let $\mathfrak{L}(n, \gamma)$ be the set of functions $f(\lambda)$ such that $(\alpha-\lambda)^{-n}f(\lambda)$ belongs to $\mathfrak{L}(0, \gamma)$ for $|\Re(\alpha)|>\gamma$. Then, if $x\in D_n(T)$, and $f\in\mathfrak{L}(n, \gamma)$, $f(T)x$ is defined by the formula

$$f(T)x=\frac{1}{2\pi i}\int_{\Gamma_c}\frac{f(\lambda)R_\lambda(T)(\alpha I-T)^n}{(\alpha-\lambda)^n}d\lambda,$$

where $\gamma<c<|\Re(\alpha)|$, and Γ_c is the contour formed by the pair of lines $\sigma=\pm c$.

The definition is consistent in the sense that

$$f(T)g(T)x=(fg)(T)x$$

when $f\in\mathfrak{L}(m, \gamma)$, $g\in\mathfrak{L}(n, \gamma)$ and $x\in D_{m+n}(T)$. It is also consistent with that given by A. E. Taylor [Acta Math. **84**, 189–224 (1951); these Rev. **12**, 717] when both of them are applicable. The operator $f(T)$ with domain $D_n(T)$ is not necessarily closed, but the author constructs its closure explicitly.

Under the additional assumptions that T has a dense domain, and that $\|R_\lambda(T)\|\leqq 1/(|\sigma|-\gamma)$, the author shows that T is the infinitesimal generator of a group of bounded operators $\mathfrak{T}(\xi)$, defined for all real ξ, and satisfying

$$\mathfrak{T}(\xi+\eta)=\mathfrak{T}(\xi)\mathfrak{T}(\eta), \quad \|\mathfrak{T}(\xi)\|\leqq e^{\gamma|\xi|}.$$

The last part of the paper is devoted to the construction of inversion formulae for the operators $f(T)$ in cases when such formulae may be expected to exist; as an application the author deduces a theorem of Hirschman and Widder [Pacific J. Math. **1**, 211–225 (1951); these Rev. **13**, 342].

F. Smithies (Cambridge, England).

Cherubino, Salvatore. Matrici e sistemi lineari infiniti. Ann. Scuola Norm. Super. Pisa (3) **6** (1952), 291–315 (1953).

This is a sequel to an earlier paper [same Ann. (3) **3**, 133–159 (1950); these Rev. **12**, 235]. It considers the solution of the infinite system of linear equations $xA=b$, x and b infinite sequences, and A a doubly infinite matrix. Consider first the case in which the rows of A are assumed to be linearly independent in the sense that $xA=0$ implies $x=0$; then the solution of the system (1) is tied up with the solution of the system (2): $xN=xA\bar{A}_{-1}=b\bar{A}_{-1}$, where $\bar{A}_{-1}$ is the conjugate transpose of A. For this it is necessary to assume that $N=A\bar{A}_{-1}$ exists, which requires the rows of A to be in Hilbert space. Further there enter convergence properties of $b\bar{A}_{-1}$ and associative properties of $xA\bar{A}_{-1}$. (1)

and (2) are then equivalent if the columns of A are also linearly independent in the author's sense. The reduction of the matrix N to a diagonal form developed in the earlier paper is used to indicate a procedure which might lead to a form of solution of (2). Since precise conditions for the validity of the limit processes involved are lacking, the results are largely heuristic or formal. There are some results on the solution of the homogeneous system $xA=0$, with $x\neq 0$. The last section is devoted to a proof of the theorem on the product of determinants as applied to infinite matrices, viz. $\det AB=\det A \det B$, a substitute for the one contained in the preceding paper. $\det A$ is defined as $\lim_n \det A_n$, where A_n is the determinant of the upper left-hand submatrix of order n by n. In addition to assuming tacitly that any convergent infinite series can be rearranged in any order, the proof also assumes without proof that if $\lim_n a_{ij_n}=a_{ij}$, then

$$\lim_n \det (a_{ij_n}; i, j=1, \cdots, n)=\det (a_{ij}; i, j=1, \cdots),$$

raising reasonable doubts of its validity.

T. H. Hildebrandt (Ann Arbor, Mich.).

Halmos, Paul R. Spectra and spectral manifolds. Ann. Soc. Polon. Math. **25** (1952), 43–49 (1953).

Let A be a bounded normal operator on a Hilbert space $\mathfrak{H}$, let $\mathfrak{J}$ be a closed invariant subspace of $\mathfrak{H}$, and let B be the restriction of A to $\mathfrak{J}$. Let $\Lambda(\mathfrak{J})$ consist of those complex numbers λ for which $B-\lambda$ is not invertible. A reducing subspace is an invariant subspace whose ortho-complement is also invariant. Since the intersection of every set of reducing subspaces is a reducing subspace there is a unique minimal reducing subspace $\mathfrak{R}$ containing a given invariant subspace $\mathfrak{J}$. It is shown that $\Lambda(\mathfrak{R})\subset\Lambda(\mathfrak{J})$. It is also shown that for Borel sets M in the complex plane the closed linear manifold spanned by all invariant subspaces $\mathfrak{J}$ for which $\Lambda(\mathfrak{J})\subset M$ is the same as the closed linear manifold spanned by all reducing subspaces $\mathfrak{J}$ for which $\Lambda(\mathfrak{J})\subset M$. *N. Dunford.*

Cordes, Heinz Otto. Separation der Variablen in Hilbertschen Räumen. Math. Ann. **125**, 401–434 (1953).

This paper seeks to replace the procedure of separation of variables in finding eigenfunctions and eigenvalues for special partial differential equations by an abstract procedure in Hilbert space utilizing the concept of direct products as developed by Murray and v. Neumann [Ann. of Math. (2) **37**, 116–229 (1936)] and later Schatten [A theory of cross-spaces, Princeton, 1950; these Rev. **12**, 186]. The first two thirds of the paper give the details of the direct product spaces and operators on them which the author considers, and which are not at all obvious for the given differential equation examples. The author's results which follow are a number of criteria for essential self-adjointness of his final operator; these may be a bit clumsy to apply. In a future paper the author intends to use these criteria to establish completeness of the resulting eigenfunction products for various examples, which he claims will give stronger results than those already established by standard separation of variables. The significance of the present paper with its new approach will attend the success of this future one. There are a few confusing typographical errors in some of the definitions of the present paper, and, more annoying, the interchange of the terms "complete" and "closed". *F. H. Brownell* (Princeton, N. J.).

Sz.-Nagy, Béla. A moment problem for self-adjoint operators. Acta Math. Acad. Sci. Hungar. **3** (1952), 285–293 (1953). (Russian summary)

R. V. Kadison [Ann. of Math. (2) **56**, 494–503 (1952); these Rev. **14**, 481] obtained a result which he called the generalized Schwarz inequality. The present author gives a quite different kind of proof for the following somewhat more general formulation of the Kadison result. Let X be a compact subset of the reals and let $\{A_k\}$ $(k=0, 1, 2, \cdots)$ be a sequence of bounded self-adjoint operators on the Hilbert space H which satisfy the two conditions; (α_X) If $c_0+c_1x+\cdots+c_nx^n\geqq 0$ for $x\in X$, then

$$c_0A_0+c_1A_1+\cdots+c_nA_n\geqq 0,$$

and (β) $\|A_0\|\leqq 1$. Then $A_1^2\leqq A_2$. Furthermore, in order that $A_1^2=A_2$ it is necessary and sufficient that $A_k=A_1^k$ $(k=1, 2, \cdots)$. The sequences $\{A_k\}$ which satisfy the condition (α_X) of the above theorem are characterized as follows: A_0 can be any non-negative bounded operator on H. Let H_+ be any Hilbert space containing H as a (not necessarily proper) subspace and let A_+ be any bounded self-adjoint operator on H_+ whose spectrum is contained in X. Then $A_k=A_0^{\frac{1}{2}}P_+A_+^kA_0^{\frac{1}{2}}$ where P_+ is the projection of H_+ on H. As a corollary one obtains that a sequence $\{A_k\}$ of bounded self-adjoint operators on H is totally monotonic (i.e., $\sum_{k=0}^{n}(-1)^k\binom{n}{k}A_{m+k}\geqq 0$, for all m, n) if and only if $A_k=A_0^{\frac{1}{2}}P_+A_+^kA_0^{\frac{1}{2}}$ where the spectrum of A_+ is contained in $[0, 1]$. *C. E. Rickart* (New Haven, Conn.).

Nakamura, Masahiro, and Takeda, Zirô. The Radon-Nikodým theorem of traces for a certain operator algebra. Tôhoku Math. J. (2) **4**, 275–283 (1952).

Let τ be a trace of the W^* algebra R of operators in Hilbert space H. It is assumed that R and τ satisfy one of the following: (i) A set of mutually orthogonal projections of R is at most countable and τ is countably additive, i.e., $\tau(\sum_{i=1}^{\infty}p_i)=\sum_{i=1}^{\infty}\tau(p_i)$ whenever p_i is a set of mutually orthogonal projections; (ii) H is separable and τ is countably additive; (iii) τ is continuous in the order determined by the positive elements xx^*. Let σ be another trace of R with $0\leqq\sigma\leqq\kappa\tau$ where κ is a constant. Then there is an essentially unique positive element a in the center of R which has the properties; $0\leqq a\leqq\kappa$, and, for an arbitrary element x in R, $\sigma(x)=\tau(xa)$. The element σ has the integral representation $\sigma(x)=\int\chi(x)\chi(a)d\tau(\chi)$ where the integration is over the character space and $d\tau$ is the regular measure determined by τ. An extension of this theorem to the case where the absolute continuity of σ with respect to τ replaces the assumption $0\leqq\sigma\leqq\kappa\tau$ is also given. *N. Dunford.*

Blum, Edward K. The fundamental group of the principal component of a commutative Banach algebra. Proc. Amer. Math. Soc. **4**, 397–400 (1953).

Let B denote a commutative complex Banach algebra with unit element e (and $\|e\|=1$), and let $\exp(x)$ denote the exponential function defined on B by the classical infinite series. Let G_1 be the maximal subset of B with the properties: (1) G_1 is open and connected; (2) G_1 contains e and all the elements of G_1 have multiplicative inverses. The set G_1 is a multiplicative group. The main theorem of the paper is that if P is the set of all x in B such that $\exp(x)=e$, then P is an additive group isomorphic to the fundamental group of G_1. Two proofs are given. For one proof the author considers closed curves in G_1 beginning and ending at e, and demonstrates that within the class of such curves which are all homotopic to a particular one there is a unique curve

of the form $\exp(tb)$, $0 \leqq t \leqq 1$, where $\exp(b) = e$. The argument utilizes Cauchy's integral theorem, the validity of which for commutative Banach algebras was verified by E. R. Lorch [Trans. Amer. Math. Soc. 54, 414–425 (1943); these Rev. 5, 100]. The other proof is based on a theorem of Schreier occurring in the theory of the universal covering group. To apply this theorem it is necessary to show that the elements of P are isolated. This is demonstrated by spectral theory arguments. *A. E. Taylor.*

Koopman, B. O. Exponential limiting products in Banach algebras. Trans. Amer. Math. Soc. 70, 256–276 (1951).

Let $\mathfrak{R} = \{a\}$ be a (not necessarily commutative) Banach algebra with the unit element 1 and the norm $N(a)$. The exponential function in $\mathfrak{R}$ is defined by

$$\exp(a) = 1 + a/1! + a^2/2! + \cdots.$$

The author proves the following generalizations of the formula of elementary calculus $\lim_{n\to\infty}(1+a/n)^n = \exp(a)$ in $\mathfrak{R}$: Let $\{\omega_n \mid n = 1, 2, \cdots\}$ be an increasing sequence of positive numbers such that $\omega_n \to \infty$ and $\omega_{n+1}/\omega_n \to 1$. If $\{a_n \mid n = 1, 2, \cdots\}$ is a sequence of elements from $\mathfrak{R}$ such that $\lim_{n\to\infty} \omega_n^{-1} \sum_{k=1}^{n} a_k = m$ for some $m \in \mathfrak{R}$ and

$$\sup_n \omega_n^{-1} \sum_{k=1}^{n} N(a_k) < \infty,$$

then $\lim_{n\to\infty} \prod_{k=1}^{n}(1+a_k/\omega_n) = \exp(m)$. Further, if

$$\{c_{n,k} \mid k = 1, \cdots, n;\ n = 1, 2, \cdots\}$$

is an infinite triangular array of elements from $\mathfrak{R}$ such that

$$\lim_{n\to\infty} \omega_n^{-1} \sum_{k=1}^{n} N(c_{n,k}) = 0,$$

then

$$\lim_{n\to\infty} \prod_{k=1}^{n}[1 + (a_k + c_{n,k})/\omega_n] = \exp(m).$$

Also, (i) if $\{b_n \mid n = 1, 2, \cdots\}$ is a sequence of elements from $\mathfrak{R}$ such that $\sup_n N(b_n) < \infty$ and $\lim_{n\to\infty} b_k \cdots b_{k+n} = B_k$ for some $B_k \in \mathfrak{R}$ uniformly in k; (ii) if $\{g_n \mid n = 1, 2, \cdots\}$ is a sequence of elements from $\mathfrak{R}$ such that $\sup_n \omega_n^{-1} \sum_{k=1}^{n} N(g_k) < \infty$ and $\lim_{n\to\infty} g_n/\omega_n = 0$; and (iii) if $\lim_{n\to\infty} \omega_n^{-1} \sum_{k=1}^{n} g_k B_{k+1} = M$ for some $M \in \mathfrak{R}$, then $\lim_{n\to\infty} \prod_{k=1}^{n}(b_k + g_k/\omega_n) = B_1 \exp(M)$. These results are used by the author [same vol., 277–290 (1951); these Rev. 14, 1100] to prove a law of small numbers in Markoff chains. *S. Kakutani* (New Haven, Conn.).

Ryll-Nardzewski, C. Sur la convergence des séries de puissances de l'opérateur différentiel. Studia Math. 13, 37–40 (1953).

L'auteur se place dans le cadre du calcul opératoire de Mikusiński [Studia Math. 11, 41–70 (1949); ces Rev. 12, 189] (passage au corps des quotients de l'anneau d'intégrité défini par la convolution sur les fonctions continues sur $(0, +\infty)$; introduction sur ce corps d'une notion de convergence). Il montre ceci: s désignant l'opérateur différentiel (appelé aussi p), la série $\sum_{n=0}^{\infty} \gamma_n \lambda^n s^n$ converge pour tout λ complexe s'il existe $\delta > 1$ tel que $\lim\sup_{n\to\infty} (n^n)^{\delta} |\gamma_n| < \infty$, et diverge pour tout λ complexe $\neq 0$ si $\lim\sup_{n\to\infty} n^n |\gamma_n| > 0$. Il en résulte qu'on peut écrire $\exp(s^{\alpha}\lambda) = \sum_{n=0}^{\infty} s^{n\alpha} \lambda^n / n!$, pour tout λ complexe, si $\alpha < 1$. *L. Schwartz* (Paris).

Ryll-Nardzewski, C. Sur les séries de puissances dans le calcul opératoire. Studia Math. 13, 41–47 (1953).

Soit μ une mesure de Radon portée par la demi-droite $(0, +\infty)$, $\|\mu\|_A = \int_0^A |d\mu|$, μ^{*n} le produit de convolution de n mesures égales à μ. Théorème: Pour tout $A > 0$, $\lim_{n\to\infty} \|\mu^{*n}\|_A^{1/n} = |\mu_0|$, où μ_0 est la masse de μ localisée à l'origine. Démonstration. On a $\mu = \mu_1 + \mu_2$, où $\int_0^{+\infty} |d\mu_1| = \theta$, $\theta > |\mu_0|$ arbitraire, et où μ_2 a son support dans $(a, +\infty)$, $a > 0$. Soit p entier tel que $pa > A$. Pour $\nu \geqq p$, $\mu_2^{*\nu}$ a son support dans $(pa, +\infty)$ donc est nulle dans $(0, A)$, de sorte que $\mu^{*n} = \sum_{\nu=0}^{p} \binom{n}{\nu} \mu_1^{*n-\nu} \mu_2^{*\nu}$ dans $(0, A)$, et

$$\|\mu^{*n}\|_A \leqq C n^p \theta^{n-p},$$

d'où $\lim\sup_{n\to\infty} \|\mu^{*n}\|_A^{1/n} \leqq |\mu_0|$. Mais μ^{*n} porte à l'origine la masse μ_0^n, donc $\lim\inf_{n\to\infty} \|\mu^{*n}\|_A^{1/n} \geqq |\mu_0|$ cqfd. Corollaire: Dans l'algèbre de convolution des mesures de Radon portées par $(0, +\infty)$, pour qu'une mesure soit inversible, il faut et il suffit qu'elle porte à l'origine une masse $\neq 0$. Ce résultat étend beaucoup ceux qu'on connaissait jusqu'à présent. Regrettons seulement que l'auteur utilise l'outil démodé des fonctions à variation bornée pour définir des mesures de Radon et leur convolution. *L. Schwartz* (Paris).

Theory of Probability

***Braithwaite, Richard Bevan. Scientific explanation. A study of the function of theory, probability and law in science.** Cambridge, at the University Press, 1953. xii+376 pp. $8.00.

The book is based on the Tarner Lectures of Trinity College, Cambridge, delivered by the author in 1946. The author's purpose in the book is "to examine the logical features common to all the sciences". The word science is understood as a hierarchical deductive system in which the lowest-level generalizations from empirical facts are deduced from more general hypotheses, at a higher level. Among other things, the author is concerned with the role that mathematics played in the structural development of sciences. However, with the possible exception of the theory of probability and statistics, the mathematical disciplines are beyond the scope of the book. The titles of the eleven chapters are: I. Introductory; II. Scientific deductive systems and their representations; III. The status of the theoretical terms in science; IV. Models for scientific theories, their use and misuse; V. Statistical hypotheses, probability statements and class ratio arithmetic; VI. The meaning of probability statements within a scientific system; VII. The choice between statistical hypotheses; VIII. The justification of induction; IX. Laws of nature and causality; X. Causal and teleological explanations; XI. Explanation of scientific laws.

In the opinion of the reviewer (and this seems to coincide with the attitude of the author) the most interesting chapters are V, VI, and VII, given to probability and statistics. A book written by a Cambridge philosopher and dedicated to the memory of John Maynard Keynes might be expected to follow the lines of thought established by Keynes and broadened by Harold Jeffreys. In other words, the reviewer expected to find that the mathematical concept of probability is treated as a measure of "intensity of belief". Instead, Professor Braithwaite adopts the attitude, shared by many modern probabilists, that mathematical probability is an abstraction of empirical relative frequency. The probability of a member of a finite class β being also a member of α is defined simply as a quotient of the number of specimens $(\alpha\beta)$ divided by the number of β. Somewhat unexpectedly, the author refuses to extend this definition to infinite sets with measure taking the place of number.

A mathematician will find a number of similarly unexpected details in Professor Braithwaite's book and some of them may be difficult to accept. Thus, for example, the concept of a statistical hypothesis appears to precede that of probability. In fact, "statistical hypothesis" is introduced in distinction to "universal hypothesis". The latter has the form "all the things which are A are also B". Contrary to this a statistical hypothesis asserts that a certain proportion of things that are A are also B. This, then, leads the author to the concept of probability.

The subject of Chapter VI, viz. the meaning of probability statements within a scientific system, is hardly ever considered in the probabilistic literature and leads to interesting speculations. If a particular statement is such that it is impossible, even in principle or in our imagination, to verify whether it is true or false, then to the author this statement has no meaning. Thus, in order that a statement be meaningful, there must exist a method of verifying it. In particular, in order that a probability statement be considered meaningful, it is necessary to have a method of at least provisional verification. Considerations of this kind lead the author to the theory of testing of statistical hypotheses and to estimation, treated, so to speak, as a preliminary to probability intervening in the structure of a science. In fact, the author emphasizes that "the series of rejection tests is proposed for use in an explanation of the meaning of probability statements and is not proposed as the best method . . ." of testing a given statistical hypothesis. The problems of testing statistical hypotheses as such is treated in Chapter VII as a particular case of Wald's general theory of statistical decision functions, connected with von Neumann's theory of games. *J. Neyman.*

Ren'i, A. On the foundations of probability theory. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie I. **47**, 227–236 (1951). (Russian)

***Cantelli, Francesco Paolo. Calcolo delle probabilità e analisi matematica.** Atti del Quarto Congresso, dell' Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 17–26. Casa Editrice Perrella, Roma, 1953.

Il conferenziere si intrattiene sui contributi apportati all'Analisi Matematica dal Calcolo delle Probabilità, e ne fornisce degli esempi. *Author's summary.*

Dufresne, Pierre. Problèmes de dépouillements. II, III. Problèmes intéressant un nombre non limité de candidats. Gaz. Mat., Lisboa **11**, no. 46, 6–12 (1950); **12**, no. 47, 13–15 (1951).

Dufresne, Pierre. Problèmes de dépouillements. IV, V. Triangles imités du triangle arithmétique de Pascal. Gaz. Mat., Lisboa **13**, no. 52, 3–6 (1952); **14**, no. 54, 14–17 (1953).

Redheffer, R. M. A note on the Poisson law. Math. Mag. **26**, 185–188 (1953).

Assuming points distributed on the real line such that only a finite number occur in any finite interval and the probability that an interval dx contains n points is independent of the number in any other nonoverlapping interval, the author shows by the use of generating functions that the probability of n points in an interval of given length is a mixture of Poisson distributions. *J. L. Snell.*

Skitovič, V. P. On a property of the normal distribution. Doklady Akad. Nauk SSSR (N.S.) **89**, 217–219 (1953). (Russian)

If $X_1, \cdots, X_n$ are independent random variables and if there exist two linear forms $\sum_1^n a_i X_i$ and $\sum_1^n b_i X_i$ with $a_i b_i \neq 0$ for $1 \leqq i \leqq n$ which are independent, then all the X_i are normally distributed. This result subsumes all the previous known ones in this direction. The proof depends on forming differences of the logarithmic characteristic functions and applying a theorem of Marcinkiewicz [Math. Z. **44**, 612–618 (1938)]. *K. L. Chung* (Syracuse, N. Y.).

Girault, Maurice. Application du produit de composition aux fonctions caractéristiques. Démonstration d'un théorème de M. Khintchine. C. R. Acad. Sci. Paris **237**, 20–22 (1953).

The function

$$\phi_3(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_1(t+\theta)\bar{\phi}_2(\theta)d\theta\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\phi_1(t+\theta)\bar{\phi}_2(\theta)d\theta\right)$$

where $\phi_1(t)$ and $\phi_2(t)$ are characteristic functions corresponding to the probability densities $f_1(x)$ and $f_2(x)$ (probability jumps $p_1(x)$ and $p_2(x)$) is the characteristic function corresponding to the probability density $f_3(x) = f_1(x)f_2(x)$ (probability jumps $p_3(x) = p_1(x)p_2(x)$). Application to the Wiener-Khintchine theorem on characteristic functions. [See also Bochner, Math. Ann. **108**, 378–410 (1933), in particular, relation (6.43).] *M. Loève.*

***Prékopa, András. Sur un problème du calcul des probabilités.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 759–764. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and French summaries)

One of two flasks of water is chosen at random and a random quantity of water is poured out of it. If this procedure is repeated until one of the flasks is empty, what is the expected amount of water remaining in the other flask? The answer is written down in terms of the probability of choosing a particular one of the flasks and the distribution function of the random pourings. The question of several flasks is also discussed. The problem is the continuous generalization of the Banach match-box problem.

P. R. Halmos (Chicago, Ill.).

Fraser, D. A. S. Generalized hit probabilities with a Gaussian target. II. Ann. Math. Statistics **24**, 288–294 (1953).

See the author's "Generalized hit probabilities with a Gaussian target" [Princeton, 1949; these Rev. **11**, 257]. *J. L. Doob* (Urbana, Ill.).

Dynkin, E. B. Classes of equivalent random quantities. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 125–130 (1953). (Russian)

The author generalizes results of de Finetti and Khintchine on equivalent chance variables. The chance variables $x_1, x_2, \cdots$ are called equivalent if, for any integral k and any k integers $i_1 < i_2 < \cdots < i_k$,

$$P\{x_{i_1} < y_1, x_{i_2} < y_2, \cdots, x_{i_k} < y_k\} = F_k(y_1, \cdots, y_k),$$

where F_k depends only on k and not on $i_1, \cdots, i_k$. The author proves the following characterization of the functions F_k for any k: Let $G(y, \theta)$ be a distribution function in y for almost all points θ of a probability space Ω with probability

measure π. Then

$$F_k(y_1, \cdots, y_k) = \int_\Omega \prod_{i=1}^{k} G(y_i, \theta) d\pi(\theta)$$

for some G, Ω, and π. The converse is obviously true. Some measurability conditions should be appended.

J. Wolfowitz (Ithaca, N. Y.).

Nisnevič, L. B. On the scheme of Markov's urns. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 131–134 (1953). (Russian)

Making use of the theorem of Dynkin [see the previous review] the author gives an urn scheme by which one can realize the probability distribution of any sequence of equivalent chance variables which take only the values $1, \cdots, m$. *J. Wolfowitz* (Ithaca, N. Y.).

***Kolmogorov, A. N. A theorem on the convergence of conditional mathematical expectations and some of its applications.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 367–386. Akadémiai Kiadó, Budapest, 1952. (Russian and Hungarian)

The author proves the theorem that (roughly) the conditional expectation of a random variable relative to the nth of an increasing sequence of conditions converges in the mean of order 1 as $n \to \infty$. This theorem is then applied to furnish a simple proof of a theorem of Rényi [Acta Math. Acad. Sci. Hungar. **1**, 99–108 (1950); these Rev. **12**, 619]. He is evidently unaware that the theorem on conditional expectations, which is even true with probability 1 convergence, was proved by the reviewer [Trans. Amer. Math. Soc. **47**, 455–486 (1940); these Rev. **1**, 343] as a slight generalization of a theorem of Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937].

J. L. Doob (Urbana, Ill.).

Marušin, M. N. Proof of S. N. Bernštein's generalized fundamental lemma for sums of almost independent quantities satisfying Lindberg's condition. Doklady Akad. Nauk SSSR (N.S.) **90**, 21–24 (1953). (Russian) Content given in title. *K. L. Chung.*

***Gyires, Béla. Über den Grenzwert von Summenverteilungen.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 741–758. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

The paper deals with limit distributions of sequences of mutually independent, integer-valued random variables. In the first part the limit distribution of sums mod n is discussed while the general case is treated in the second part.

E. Lukacs (Washington, D. C.).

Smith, Walter L. A frequency-function form of the central limit theorem. Proc. Cambridge Philos. Soc. **49**, 462–472 (1953).

Let $x_1, x_2, \cdots$ be random variables which are mutually independent and have a common distribution with mean 0 and variance 1. Suppose also (*) that the distribution has characteristic function which is $O(|t|^a)$ at ∞, for some $a<0$. Let h_n be the density of distribution (if there is one) of $n^{-1/2}\sum_1^n x_j$. The author proves that h_n exists for sufficiently large n, and that, for $0 \leqq m \leqq 2$,

$$|x|^m h_n(x) \to (2\pi)^{-1/2} |x|^m \exp(-x^2/2)$$

uniformly in the variable x. This result is slightly more general than that given by Gnedenko and Kolmogorov [Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949; these Rev. **12**, 839]. If the distribution of x_j has a finite rth moment ($r \geqq 2$) the above result holds for $m \leqq r$. The condition (*) can be replaced by the condition that the distribution function of x_j has a derivative equal almost everywhere on some interval to a positive function of bounded variation. A version of the central limit theorem for densities is then proved without the hypothesis of a common distribution of the x_j's, under a suitable strengthening of the Lindeberg condition. A result is also proved on the L_1 convergence of the density functions involved to the above limit. For such theorems see also Prohorov [Doklady Akad. Nauk SSSR (N.S.) **83**, 797–800 (1952); these Rev. **14**, 186]. *J. L. Doob.*

Lévy, Paul. Random functions: General theory with special reference to Laplacian random functions. Univ. California Publ. Statist. **1**, 331–390 (1953).

Somewhat more detailed exposition of results presented by the author at the Second Berkeley Symposium on Mathematical Statistics and Probability [see the Proceedings thereof, 1950, Univ. of California Press, 1951, pp. 171–187; these Rev. **13**, 476]. *J. L. Doob.*

Chung, Kai Lai. Contributions to the theory of Markov chains. J. Research Nat. Bur. Standards **50**, 203–208 (1953).

The random variables X_n, $n=0, 1, \cdots$, form a discrete Markov chain with states $i, j, \cdots$ and stationary transition probabilities $P_{ij}^{(n)} = P(X_{m+n}=j \mid X_m = i)$. Let

$$F_{ij}^{(n)} = P(X_n = j, X_r \neq j, 1 \leqq r < n \mid X_0 = i).$$

The subscript k prefixed to a symbol means that the state k is not visited during the passage, exclusive of both terminals. Q^* means $\sum_{n=1}^{\infty} Q^{(n)}$. The author proves a large number of relations of which only some can be given here. (1) $\lim_{N\to\infty} \sum_{n=0}^{N} P_{ii}^{(n)} / \sum_{n=0}^{N} P_{jj}^{(n)}$ equals any of the three expressions

$$(1+{}_jP^*_{ii})/(1+{}_iP^*_{jj}),\ F^*_{ij}/{}_iP^*_{ij},\ (F^*_{ij})({}_jF^*_{ji})/(F^*_{jj})({}_iF^*_{ij}),$$

the first always, the second if $i \neq j$, the third if $F^*_{ij}F^*_{ji} > 0$. Doeblin proved the existence of the limit [Bull. Soc. Math. France **66**, 210–220 (1938)]; the expressions for it are new. (2) Suppose the states form a single positive recurrent class. If $j \neq k$, then

$$m(i, j \cup k) = m_{ij} - {}_jF^*_{ik} m_{kj}, \quad m(i, j \cap k) = m_{ij} + {}_kF^*_{ij} m_{jk},$$

where m_{ij}, $m(i, j \cup k)$, and $m(i, j \cap k)$ are the mean first-passage times from i to j, i to j or k, and i to both j and k; then $m_{ik} + m_{kj} - m_{ij} = {}_kF^*_{ij}(m_{jk} + m_{ki})$. As an application certain quantities involved in the central limit theorem for Markov chains are evaluated. (3) Formulas analogous to Poincaré's formula [Feller, Introduction to probability theory . . . , Wiley, New York, 1950, p. 61; these Rev. **12**, 424] are given for mean first-passage times to unions or intersections of an arbitrary number of states. (4) If i, j, and k (not necessarily distinct) belong to the same positive class, $\sum_{n=1}^{\infty}(P_{ik}^{(n)} - P_{jk}^{(n)}) = (m_{jk} - m_{ik})/m_{kk}$. There are some connections with work of the reviewer [Trans. Amer. Math. Soc. **73**, 471–486 (1952); these Rev. **14**, 567].

T. E. Harris (Santa Monica, Calif.).

Koopman, B. O. A law of small numbers in Markoff chains. Trans. Amer. Math. Soc. **70**, 277–290 (1951).

Let $\{U_{n,k} | k=1, 2, \cdots, n; n=1, 2, \cdots\}$ be an infinite triangular array of random variables taking only the values 0 and 1 such that, for each n, the row $\{U_{n,k} | k=1, \cdots, n\}$ forms a Markoff chain. Put

$$p_{n,k}=\Pr\{U_{n,k}=1\},\quad a_{n,k}=\Pr\{U_{n,k}=1 | U_{n,k-1}=1\},$$
$$b_{n,k}=\Pr\{U_{n,k}=1 | U_{n,k-1}=0\}.$$

Then $p_{n,k}$ $(k=2, \cdots, n)$ are determined by the recurrence formula $p_{n,k}=a_{n,k}p_{n,k-1}+b_{n,k}(1-p_{n,k-1})$, and hence by $p_{n,1}$ and $a_{n,i}$, $b_{n,i}$ $(i=2, \cdots, k)$. It is assumed that the transition probabilities $a_{n,k}$ and $b_{n,k}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} p_{n,1}=0$ implies $\lim_{n\to\infty} \max_{1\leq k\leq n} p_{n,k}=0$;

(ii) $\lim_{n\to\infty} p_{n,1}=p_1$ implies $\lim_{n\to\infty} \sum_{k=1}^{n} p_{n,k}=m$;

(iii) $\lim_{n\to\infty} p_{n,1}=p_1$ implies

$$\lim_{n\to\infty} \sum_{k=0}^{n-\nu} \Pr\{U_{n,k}=1, U_{n,k+\nu}=1\}=R_\nu.$$

Let us further put $P_n(s)=\Pr\{U_{n,k}=1 \text{ } s \text{ times when } k \text{ runs through } 1, \cdots, n\}$, $s=0, \cdots, n$; $n=1, 2, \cdots$. Then the expected total number of 1's in the nth row is given by $\sum_{k=1}^{n} p_{n,k}=\sum_{s=0}^{n} sP_n(s)$. The author proves that the limit $\lim_{n\to\infty} P_n(s)=P(s)$ exists under the following additional conditions: (iv) $\lim_{n\to\infty} a_{n,k}=a_k$ $(\leqq a<1)$ exists uniformly in k; and (v) there exists a sequence $\{c_k | k=2, 3, \cdots\}$ of non-negative numbers not all equal to zero, such that $\lim_{n\to\infty} e_{n,k}=0$ uniformly in k if $e_{n,k}$ is defined by

$$b_{n,k}\left[\sum_{i=2}^{n} b_{n,i}\right]^{-1}=c_k\left[\sum_{i=2}^{n} c_i\right]^{-1}(1+e_{n,k}).$$

The author also shows that the generating function

$$\phi(t)=\sum_{s=0}^{\infty} P(s)t^s$$

of the limiting distribution $P(s)$ can be expressed as $\phi(t)=[1-f(t)]\exp\{-(1-t)[m-(1-t)\psi(t)]+f(t)\}$, where $\psi(t)=\sum_{r=1}^{\infty} R_r t^{r-1}$ and $f(t)=(1-t)p_1\sum_{r=0}^{\infty} a_1\cdots a_{r+1}t^r$. The proof is based on a previous result of the author [same vol., 256–276 (1951); these Rev. **14**, 1097] concerning exponential limiting products in Banach algebras.

The case when each row forms an independent system, i.e., the case when $p_{n,k}=a_{n,k}=b_{n,k}$, $k=2, \cdots, n$, was previously discussed by the author [Proc. Amer. Math. Soc. **1**, 813–823 (1950); these Rev. **12**, 424]. In this case $P(s)$ is a Poisson distribution, i.e., $P(s)=e^{-m}m^s/s!$, $s=0, 1, \cdots$, and it was shown that (i), (ii) together form a necessary and sufficient condition for $\lim_{n\to\infty} P_n(s)=e^{-m}m^s/s!$, $s=0, 1, \cdots$. Also, the case when each row forms a stationary system, i.e., the case when $a_{n,k}=a_n$, $b_{n,k}=b_n$, $k=2, \cdots, n$, was discussed by the author [Proc. Nat. Acad. Sci. U. S. A. **36**, 202–207 (1950); these Rev. **11**, 445] and it was shown that $P(s)$ is a Laguerre distribution. *S. Kakutani.*

Ramakrishnan, Alladi. Stochastic processes associated with random divisions of a line. Proc. Cambridge Philos. Soc. **49**, 473–485 (1953).

A class of stochastic processes associated with points randomly distributed in a line of finite extension L is considered. A general integral equation for the function representing the probability distribution of the stochastic variable under consideration is derived and solved by using the Laplace transform technique . . . the problem of the fluctuations in brightness of the Milky Way is discussed. The results of Chandrasekhar and Münch are derived. . . . (From the author's abstract.) *K. L. Chung.*

Florek, K., Marczewski, E., and Ryll-Nardzewski, C. Remarks on the Poisson stochastic process. I. Studia Math. **13**, 122–129 (1953).

The authors prove that a temporally homogeneous stochastic process with independent increments is Poisson if and only if one is the probability of the set of all integral-valued functions which are continuous on the right, non-decreasing, and have only jumps equal to one.

J. Wolfowitz (Ithaca, N. Y.).

Marczewski, E. Remarks on the Poisson stochastic process. II. Studia Math. **13**, 130–136 (1953).

Let Ω_0 be the set of all integral-valued functions $\omega(t)$ defined for $t\geqq 0$ which are continuous on the right, non-decreasing, and such that $\omega(0)=0$. Let (Ω, B_Ω, μ) be a stochastic process, where $\Omega\subset\Omega_0$, B_Ω is the smallest Borel field over sets of ω defined by inequalities at a finite number of values of t, and μ a probability measure on B_Ω. Let I be the generic designation of a t-interval closed on the right and open on the left, and let $d_I(\omega)$ be the increment of ω on I. Let b be the generic designation of a Borel set on the t-line, and $d_b(\omega)$ be the sum of the jumps of ω over all points t in b. Suppose: (h) the distribution function of the random variable d_I depends only on the length of I; (i) if $I_1, \cdots, I_n$ are disjoint then the random variables $d_{I_1}, \cdots, d_{I_n}$ are independently distributed. The author proves: I) the distribution function of d_b depends only upon the measure of b, for any b; II) if the Borel sets b_j converge in (Lebesgue) measure to b, then d_{b_j} converges in probability (according to μ) to d_b; III) if $b_1, \cdots, b_n$ are disjoint Borel sets then $d_{b_1}, \cdots, d_{b_n}$ are independently distributed. *J. Wolfowitz.*

Saxén, Tryggwe. Sur les mouvements aléatoires et le problème de ruine de la théorie de risque collective. Soc. Sci. Fenn. Comment. Phys.-Math. **16**, no. 2, 55 pp. (1951).

Let $\{x(t), t\geqq 0\}$ be a stochastic process with stationary independent increments, $x(0)=0$, and suppose that the logarithm of the characteristic function of $x(t)$ is

$$ct\int_{-\infty}^{\infty}(e^{isa}-1-isa)dQ(a),$$

where c is a constant, $Q(a)$ is a distribution function with mean m, $Q(a)=0$ for $a<0$, and s is the argument of the characteristic function. Let $f(t)$ and $f_1(t)$ be two continuous, monotonically increasing functions, $f(0)<0$, $f_1(0)>0$. The author considers two problems: A) to find the probability $\psi(T)$ of the event $\{x(t)<f(t)-cmt$ for some $t<T\}$; B) to find the probability $\psi_1(T)$ of the event $\{x(t)>f_1(t)-cmt$ for some $t<T\}$. Let $z(\eta, \tau; T)$ be the conditional probability that, for some t in the open interval (τ, T), $x(t)<f(t)-cmt$, given that $x(\tau)=\eta-cm\tau$. It is shown that $z(\eta, \tau; T)$ satisfies an integral-differential equation

$$\frac{\partial z(\eta, \tau; T)}{\partial \tau}=c\left\{z(\eta, \tau; T)-\int_0^{\infty} z(\eta+y, \tau; T)dQ(y)\right.$$

with a unique bounded solution for the appropriate boundary conditions. The author obtains this solution as an infinite series of terms each of which is obtained by integration

from its predecessors. A similar situation obtains for problem B. These results are then applied to obtaining the probability of ruin of an insurance company. The application is illustrated by an example. *J. Wolfowitz.*

Bailey, Norman T. J. The total size of a general stochastic epidemic. Biometrika **40**, 177–185 (1953).

An epidemic in a population of n individuals [composed of x susceptibles, y infectious cases, and z removed from the population $(x+y+z=n)$] is assumed to be such that the probability of a new infection taking place in time dt is $xy\,dt$ (using an appropriate time scale) while the probability of a removal in time dt is $\rho\,dt$. Let $p_{r,s}(t)$ be the probability that at time t there are r susceptibles and s infectious individuals in the population. The author obtains the partial differential equation of the probability generating function of the $p_{r,s}$ and by Laplace transform methods is able to determine recurrence relations for $P_w=\lim_{t\to\infty} p_{n-w,0}(t)$, i.e., the total size of the epidemic. Diagrams are given showing the distribution of P_w for $n=10$, 20, and 40 and for a suitable range of values of ρ. In addition, P_w is tabulated as a function of ρ for $n=1$ to 5. From these the maximum likelihood equations are set down by which ρ could be estimated, from say the household distribution of cases of disease.

D. G. Chapman (Seattle, Wash.).

Ríos, S. Some probability laws and stochastic processes which reduce to a general type of Laplace-Stieltjes. Revista Mat. Hisp.-Amer. (4) **13**, 112–119 (1953). (Spanish)

Let $f(s,x)=\int_{-\infty}^{\infty} e^{\lambda s}\,dA(\lambda)$, where s is a real parameter. The author works out, in terms of $f(s,\infty)$, the characteristic function and moments corresponding to the cumulative distribution function $f(s,x)/f(s,\infty)$, and notes that various well-known families of distributions are reducible to this form. *H. P. Mulholland* (Birmingham).

Itô, Kiyosi. On the stochastic integral. Sûgaku **1**, 172–177 (1948). (Japanese)

This is a summary of an invited address given at a meeting of the Mathematical Society of Japan in November 1946. The author introduces the notion of a stochastic integral which is more general than that of N. Wiener and P. Lévy, and discusses fundamental properties of this stochastic integral. The details of his arguments can be found in two papers (in English) of the author [Proc. Imp. Acad. Tokyo **20**, 519–524 (1944); Proc. Japan Acad. **22**, nos. 1–4, 32–35 (1946); these Rev. **7**, 313; **12**, 191]. *S. Kakutani.*

Foster, F. G., and Good, I. J. On a generalization of Pólya's random-walk theorem. Quart. J. Math., Oxford Ser. (2) **4**, 120–126 (1953).

The results in §§1–2 are not only "similar" to previous results by Chung and Fuchs [Mem. Amer. Math. Soc. no. 6 (1951); these Rev. **12**, 722], but particular cases of them. In fact, the main result (2.5) is partly stated explicitly in Corollary to Theorem 4 in loc. cit. and partly trivially implied by (1.3) there by letting $\rho\to 1$, permissible if ϕ is real. Of course the proof is simpler in this discrete and symmetrical case. This as well as §3 goes without saying. If $f(x)$ is the probability of a displacement x at each step, $F(r)=\sum_r^s f(x)$, $V(r)=\sum_r^r x F(x)$, $v(r)=\sum_1^r x^2 f(x)$, it is shown that $\sum_1^\infty (V(r))^{-1}=\infty$ is sufficient, and $\sum_N^\infty (v(r))^{-1}=\infty$ for every N is necessary for recurrence. The conjecture that these conditions are equivalent is immediately disproved by Fuchs by taking $f(x)=0$ for $x\neq a_v$ and $f(a_v)=6/\pi^2 v^2$ where a_v, $v\geqq 1$, increases to infinity sufficiently fast.

K. L. Chung (Syracuse, N. Y.).

***Takács, Lajos. Wahrscheinlichkeitstheoretische Behandlung von Koinzidenz-Erscheinungen, mit Ereignissen gleicher Zeitdauer.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 731–740. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

This lecture contains essentially the results which the author published elsewhere [Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 371–386 (1951); Acta Math. Acad. Sci. Hungar. **2**, 275–298 (1951); these Rev. **13**, 956; **14**, 388]. *E. Lukacs* (Washington, D. C.).

Bochner, S. Fourier transforms of time series. Proc. Nat. Acad. Sci. U. S. A. **39**, 302–307 (1953).

Suppose that $F(I)$ is a random variable corresponding to the semiclosed interval I. Then F is called a Khintchine function if $E\{F(I_1)\bar{F}(I_2)\}$ is invariant under translation of the intervals, is called orthogonal if the above expectation vanishes when the intervals have no common point, is called orthonormal if it is orthogonal and also is a Khintchine function. Then, formally at least, to each Khintchine function corresponds an orthogonal function and conversely, under an integrated Fourier transform. Known special cases of this correspondence are (A) the spectral representation of a stationary process and (B) the transform taking an orthonormal function into an orthonormal function. The author connects these two cases by discussing families $\{f(\epsilon,x), 0<\epsilon<\infty\}$ of Khintchine point functions (=stationary stochastic processes) which can be put in the form $f(\epsilon,x)=\int_{-\infty}^{\infty}\exp(2\pi i x y-2\epsilon\pi y^2)\,dG(y)$, where G is an orthogonal function. A family has this form if a certain semigroup requirement is imposed on the dependence on ϵ. A Khintchine point function generates such a family which reduces to the point function when $\epsilon\to 0$, and an orthonormal function also generates such a family which reduces to the given function in an extended sense when $\epsilon\to 0$.

J. L. Doob (Urbana, Ill.).

Gihman, I. I. On the theory of differential equations of random processes. II. Ukrain. Mat. Žurnal **3**, 317–339 (1951). (Russian)

The author continues his proofs of results announced earlier [Doklady Akad. Nauk SSSR (N.S.) **58**, 961–964 (1947); these Rev. **9**, 293]. In his first paper [Ukrain. Mat. Žurnal **2**, no. 4, 37–63 (1950); these Rev. **14**, 61] he solved the equation $dx(t)=da[t,x(t)]$ to find a stochastic process with variables $\{x(t)\}$, given the a process of a rather general type. In the present paper he investigates the dependence of the solution on the initial conditions. He obtains results strong enough to derive the backward diffusion equation for $E\{f[x(t)]\,|\,x(s)=y\}$, the usual parabolic equation in s, y, for sufficiently regular f, and the a process specialized to make the $x(t)$ process a Markov process of diffusion type.

J. L. Doob (Urbana, Ill.).

McMillan, Brockway. The basic theorems of information theory. Ann. Math. Statistics **24**, 196–219 (1953).

The author restudies and systematizes the information theory of Shannon [Bell System Tech. J. **27**, 379–423, 623–656 (1948); these Rev. **10**, 133]. The input and output are defined as infinite sequences, and a channel is defined accordingly. The fundamental theorem for a noisy channel is

studied in some detail, its mathematical significance is made clear, and the probability limit theorems involved are carefully separated out. Following tradition, the "detailed epsilontics" of the proof of the fundamental theorem are omitted. *J. L. Doob* (Urbana, Ill.).

Mathematical Statistics

Tanner, J. C. A problem of interference between two queues. Biometrika **40**, 58–69 (1953).

The interference considered is of the kind occurring in the flow of two-directional traffic through a single lane; traffic in one direction blocks the lane to traffic in the other for a time α and to traffic in its own direction for a time β (α, β differing for the two directions). The traffic in each direction is supposed random, each α is greater than its β, and the variables determined are the average waiting times under statistical equilibrium conditions. Even with these simplifications, explicit formulas for average waiting times are not obtained, although the analysis is formally complete, and further specialization is examined. For both β's zero and both α's unity, a numerical table is given for the waiting time for various traffic densities. Another table gives numerical results for one-way blocking (both α and β for one direction zero). *J. Riordan* (New York, N. Y.).

Bhate, D. H. A note on the ratio of two non-central chi-squares. Bull. Calcutta Math. Soc. **43**, 147–148 (1951).

The author obtains the distribution and moments of u/v, when u and v are distributed in non-central χ^2 distributions with m and n degrees of freedom respectively.

L. A. Aroian (Culver City, Calif.).

Sinha, P. Distribution of total number of runs in samples from Poisson population. Calcutta Statist. Assoc. Bull. **4**, 171–172 (1953).

The author shows that the distribution of total number of runs in samples of n, n large, from a Poisson population with parameter λ is again a Poisson distribution with parameter λ. *L. A. Aroian* (Culver City, Calif.).

Bliss, C. I. Fitting the negative binomial distribution to biological data. Biometrics **9**, 176–196 (1953).

The author is interested in the fitting of discrete biological distributions to the negative binomial by "efficient" methods, particularly when overdispersion occurs. Many examples are given and are compared to results when the Neyman Type A, the Thomas double Poisson, or Fisher's logarithmic distribution are used. *L. A. Aroian.*

Dumas, Maurice. L'interprétation des séries de résultats blancs et noirs. Mém. Artillerie Française **26**, 589–624 (1952).

The author is interested in whether a series of trials follows a Bernoulli probability function either with p known or unknown, and, if unknown, a method for estimating p. Let t be the probability of obtaining a run of V successes or less, $t=1-(1-p)^V$. This leads to a graphical method of fit, and if the fit is close enough, a method for estimating p when p is unknown. The reviewer believes that the chi-square test and the usual method of estimating p are more efficient and just as easy to apply. *L. A. Aroian.*

Anis, A. A., and Lloyd, E. H. On the range of partial sums of a finite number of independent normal variates. Biometrika **40**, 35–42 (1953).

The authors show that the expected value of the range of the sum of n independent normally distributed random variables is $(2/\pi)^{1/2}\sum_1^{n-1} r^{-\frac{1}{2}}$. To show this the authors evaluate the integral

$$\int_0^\infty \cdots (r) \cdots \int_0^\infty \exp\left\{\sum_1^{r-1} y_i y_{i+1} - \sum_1^r y_i^2\right\} dy_1 \cdots dy_r.$$

J. L. Snell (Princeton, N. J.).

Dwass, Meyer. On the asymptotic normality of certain rank order statistics. Ann. Math. Statistics **24**, 303–306 (1953).

Let $(R_1, \cdots, R_N)$ be a random vector which takes on each of the $N!$ permutations of the numbers $1, \cdots, N$ with equal probability $1/N!$. Let $a_{N1}, \cdots, a_{NN}$; $b_{N1}, \cdots, b_{NN}$ be real numbers defined for every N. Sufficient conditions for the asymptotic normality of $S_N = \sum_{i=1}^N a_{Ni} b_{NR_i}$ were given by Wald and Wolfowitz [same Ann. **15**, 358–372 (1944); these Rev. **6**, 163], Noether [ibid. **20**, 455–458 (1949); these Rev. **11**, 188] and Hoeffding [ibid. **22**, 558–566 (1951); these Rev. **13**, 363]. The proofs of these authors were by showing that the moments of S_N converge to the corresponding moments of the normal distribution. The author gives an alternative proof, based on a new set of assumptions which enable him to show that S_N is asymptotically distributed as a sum of N independent random variables. The author's theorem applies to cases not covered by the previous results. *W. Hoeffding* (Chapel Hill, N. C.).

Schmetterer, Leopold. Zur Bayesschen Regel. Statist. Vierteljschr. **5**, 174–178 (1952).

Elementary remarks on the use of Bayes' formula for the statistical estimation of an unknown probability. It is shown that with a large class of non-constant a priori probability densities, formulas are obtained for the a posteriori probability that are as simple as with the assumption of constant density. *B. O. Koopman* (New York, N. Y.).

Fraser, D. A. S., and Guttman, Irwin. Bhattacharyya bounds without regularity assumptions. Ann. Math. Statistics **23**, 629–632 (1952).

The procedure used by Chapman and Robbins [same Ann. **22**, 581–586 (1951); these Rev. **13**, 367] of replacing derivatives by differences in the Cramér-Rao inequality for the lower bound of the variance of an estimate of a single real parameter is here extended both through the use of higher differences and to multi-parameter situations. The results are analogous to those obtained by Bhattacharyya [Sankhyā **8**, 1–14 (1946); 201–218 (1947); these Rev. **8**, 524; **9**, 365] using higher derivatives. *D. G. Chapman.*

Kôno, Kazumasa. On inefficient statistics for measurement of dependency of normal bivariates. Mem. Fac. Sci. Kyūsyū Univ. A. **7**, 1–12 (1952).

Several functions of symmetric order statistics are constructed for estimating and testing hypotheses about the correlation coefficient of a bivariate normal population with known variances. (One of the test functions requires only that the ratio of the variances be known.) The order statistics considered are those obtained after transforming the sample into one from a bivariate normal population with independent variates. *D. M. Sandelius* (Uppsala).

Cohen, A. C., Jr. Estimating parameters in truncated Pearson frequency distributions without resort to higher moments. Biometrika **40**, 50–57 (1953).

Without resorting to higher moments as was done in earlier work by the author [Ann. Math. Statistics **22**, 256–265 (1951); these Rev. **12**, 841], the present paper treats the specific problem of estimating parameters of univariate Pearson populations from samples truncated either singly or doubly at known terminal points when the number of observations thus eliminated is unknown. Using the method of moments, estimating equations are obtained involving only the first four sample moments in contrast to the first six previously employed. The results are illustrated by a doubly truncated sample from a Type III population. In the future the author expects to treat the problem by the method of maximum likelihood. For Type III populations this has been done by Des Raj in the paper reviewed below which appeared about the same time. *L. A. Aroian.*

Des Raj. Estimation of the parameters of type III populations from truncated samples. J. Amer. Statist. Assoc. **48**, 336–349 (1953).

"The problem considered is the estimation of the mean and variance of Type III populations from singly and doubly truncated samples with known truncation points when the number of unmeasured observations is (i) unknown for each tail, (ii) known separately for each tail, (iii) known jointly for the two tails. For singly truncated samples, estimation of all three parameters has also been considered. Estimating equations are obtained by the method of moments as well as by the method of maximum likelihood. Previous results for the normal and Type III populations are particular cases of the results obtained in this paper. Numerical examples are given." (From the author's summary.)

To the list of references should be added the two further papers of Cohen [Ann. Math. Statistics **22**, 256–265 (1951); these Rev. **12**, 841; and the paper reviewed above], the latter of which appeared about the same time as the author's paper. *L. A. Aroian* (Culver City, Calif.).

Rao, C. Radhakrishna. Minimum variance estimation in distributions admitting ancillary statistics. Sankhyā **12**, 53–56 (1952).

Some remarks on various properties of unbiased estimators of a parameter θ of a density function $f(x, \theta)$ in situations where, on the basis of a random sample of n observations, a pair (U, y) (possibly vectors) is (minimal) sufficient for θ with the distribution of y independent of θ. The discussion is illustrated by an example where the maximum likelihood estimator does not have asymptotically minimum variance. *J. Kiefer* (Ithaca, N. Y.).

Rao, C. Radhakrishna. Some theorems on minimum variance estimation. Sankhyā **12**, 27–42 (1952).

Simple exposition and proofs of many ideas and results concerning minimum variance unbiased estimators (m.v.u.e.'s), many of which were obtained previously by the author and others. Emphasis is on the relationship between an estimator having the property of being a m.v.u.e. and having zero covariance with all unbiased estimators of zero, and this idea is exploited in applications to linear (quadratic) estimation of linear functions (the variance) in the Markoff set-up, and similar problems. The Koopman-Darmois form is studied and under unspecified regularity conditions is shown to include all distributions where m.v.u.e.'s exist for all sample sizes. Bhattacharya estimators are considered and a counter-example to a result stated by Davis [Ann. Math. Statistics **22**, 440–445 (1951); these Rev. **13**, 259] is given. The interesting ideas of the paper are marred by sloppy statements and proofs where regularity conditions are unstated. *J. Kiefer* (Ithaca, N. Y.).

Das, A. C. On some minimum-variance unbiassed estimates. Calcutta Statist. Assoc. Bull. **4**, 166–170 (1953).

The results of this paper coincide with those of some of the applications in the paper by Rao reviewed above. *J. Kiefer* (Ithaca, N. Y.).

van der Waerden, B. L. Testing a distribution function. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 201–207 (1953).

Let $F(x)$ be a continuous cdf, $F_n(x)$ an empirical distribution function based on n observations $x_1, \cdots, x_n$. Reject the hypothesis that $x_1, \cdots, x_n$ constitute a random sample from $F(x)$ if $\max (F - F_n)$ is too large. In particular, let $F(x)$ be the normal distribution with mean 0 and variance 1. For $n = 2, 3, 5$ and significance level .01 the power of this test is derived if under the alternative hypothesis the true distribution is normal with mean greater than 0 and variance 1. It turns out that for $n = 5$ the efficiency of the test is about .65. The author believes that for larger n the efficiency of the test is still smaller. *G. E. Noether* (Boston, Mass.).

Basu, D. On symmetric estimators in point estimation with convex weight functions. Sankhyā **12**, 45–52 (1952).

For the problem of point estimation with fixed sample-size n and where all possible c.d.f.'s $F(x_1, \cdots, x_n)$ are symmetric in the x_i and the weight function $W(F, d)$ is strictly convex (convex) in d, every admissible estimator is an (has the same risk function as some) essentially symmetric function of the x_i. Applications are given to finding good linear estimators of the mean and a quadratic estimator of the variance, of a finite population. Similar problems are considered in the Markoff set-up. *J. Kiefer.*

Kiefer, J. Sequential minimax search for a maximum. Proc. Amer. Math. Soc. **4**, 502–506 (1953).

The author considers the problem of determining an interval containing the point at which a unimodal function on the unit interval possesses a maximum. No regularity conditions are assumed and the author gives, for every $\epsilon > 0$ and every number N of values of the argument at which the function may be observed, a procedure which is ϵ-minimax among the class of all sequential non-randomized procedures which terminate by giving an interval containing the required point, where the payoff is the length of the final interval. *J. L. Snell* (Princeton, N. J.).

Schmetterer, L. Bemerkungen zum Verfahren der stochastischen Iteration. Österreich. Ing.-Arch. **7**, 111–117 (1953).

The author studies the iteration process of Robbins and Monro [Ann. Math. Statistics **22**, 400–407 (1951); these Rev. **13**, 144] which is the stochastic analogue of the one due to R. v. Mises and H. Pollaczek-Geiringer [Z. Angew. Math. Mech. **9**, 58–77 (1929)]. Under conditions slightly different from those of the reviewer [Ann. Math. Statistics **23**, 457–461 (1952); these Rev. **14**, 299] he obtains an upper bound on the second moment about the sought for point θ

of the nth approximation to θ. Convergence in the mean square is an immediate consequence. By means of this bound he is able to obtain interesting results on the order of this second moment when the sequence of coefficients is $\{n^{-\alpha}\}$, $1/2<\alpha\leqq 1$, and related sequences. For the sequence given above and $1/2<\alpha<1$ the second moment about θ of the nth approximation is less than a constant multiplied by $n^{-\alpha}$; when $\alpha=1$, the result is more complicated.

J. Wolfowitz (Ithaca, N. Y.).

Moore, P. G. A sequential test for randomness. Biometrika **40**, 111–115 (1953).

For testing between two specified simple hypotheses, in one of which a sequence of 0-or-1 random variables is independent and in the other of which it is stationary Markovian, the author suggests a sequential test based at each stage on the number of runs. [Reviewer's remark: The suggested test will not in general be admissible. The minimal complete class for a general class of problems of which this is a special case was given by the reviewer [Ann. Math. Statistics **23**, 645–646 (1952)].] *J. Kiefer.*

Ghosh, M. N. An extension of Wald's decision theory to unbounded weight functions. Sankhyā **12**, 8–26 (1952).

Under conditions too lengthy to state in detail, but essentially differing from those of Wald [Statistical decision functions, Wiley, New York, 1950; these Rev. **12**, 193; we hereafter use the terminology of this book] in no longer requiring compactness of D^t and in assuming $W(F, d^t)$ finite but unbounded, the author proves for the space D_R (R real) of decision functions with risk function uniformly less than R, the results of Wald's Theorems 3.1 and 3.2. The line of proof is similar to Wald's; in particular, it makes use of an extension of Wald's Theorem 2.15. It is also proved that there exists a minimax solution, that every minimax solution is a Bayes solution in the wide sense, and that the class of all Bayes solutions in the wide sense is complete *relative to the space $\mathfrak{D}_b$ of decision functions with bounded risk functions* [the italicized phrase is, incorrectly, not stated, but is what is proved]. Many important situations not satisfying Wald's assumptions satisfy those of the present paper.

J. Kiefer (Ithaca, N. Y.).

Rao, K. S. A simple method of deriving best critical regions similar to the sample space in tests of an important class of composite hypotheses. Biometrika **40**, 231–233 (1953).

A method is suggested for teaching the topic of the title in various standard normal examples. *J. Kiefer.*

Brownlee, K. A., Hodges, J. L., Jr., and Rosenblatt, Murray. The up-and-down method with small samples. J. Amer. Statist. Assoc. **48**, 262–277 (1953).

The method of the title for estimating the mean of a normal response curve when the variance σ^2 is known [see, e.g., Dixon and Mood, same J. **43**, 109–126 (1948); the present method differs slightly] is investigated regarding bias and mean squared error (mse) for samples of size $\leqq 10$ and various values of the error δ of the initial guess and of the jump d between trials. This is also done for a second method which offers better protection for large values of $|\delta|$. Both methods appear to be substantially better than the classical method of probit analysis, the latter requiring from 50% (when $\delta=0$) to over 300% (e.g., when $\delta/2=d=\sigma$) more observations to achieve the same mse for typical values of d and δ. *J. Kiefer* (Ithaca, N. Y.).

Lindley, D. V. Estimation of a functional relationship. Biometrika **40**, 47–49 (1953).

An attempt to clear up confusion in terminology in a controversy between J. Berkson and M. G. Kendall.

J. Wolfowitz (Ithaca, N. Y.).

Williams, E. J., and Kloot, N. H. Interpolation in a series of correlated observations. Australian J. Appl. Sci. **4**, 1–17 (1953).

Given n equally spaced observations x_i on a discrete stationary random process with observations s units apart being correlated by $\rho_s=\rho^s=e^{-\lambda s}$. Assuming ρ is known, least squares methods are given for estimating the values of members of the process midway between the observations, $x_j=\sum b_i x_i$, for the following cases: (i) no trend and a linear trend in the process; (ii) observation at points and averaged over intervals. Values of the b_i and variances of the estimates of x_j are given for $n=3, 4, 5$ and for $\rho=0(.2)1.0$. The efficiencies of three simple estimates (not dependent on a knowledge of ρ) are presented graphically for different values of ρ. If ρ is large, use $\hat{x}_j=\frac{1}{2}(x_{j-1}+x_{j+1})$; if ρ is small with no trend, use $\hat{x}_j=\sum x_i/n$; and if ρ is small with a linear trend, use $\hat{x}_j=\sum x_i/n+3\sum i x_i/n(n^2-1)$. The estimation procedures are applied to compression and density test data on maple boards. *R. L. Anderson* (Raleigh, N. C.).

Fisher, R. A. Note on the efficient fitting of the negative binomial. Biometrics **9**, 197–200 (1953).

The maximum likelihood equations for the estimation of the two parameters in the negative binomial probability function are derived and methods of solving them are discussed. *L. A. Aroian* (Culver City, Calif.).

Rijkoort, P. J., and Wise, M. E. Simple approximations and nomograms for two ranking tests. Nederl. Akad. Wetensch. Proc. Ser. A. **56**=Indagationes Math. **15**, 294–302 (1953).

"The two tests are (1) the m ranking test, as developed by Kendall and Smith [Ann. Math. Statistics **10**, 275–287 (1939); these Rev. **1**, 23] for the significance of the agreement between m different rankings of a sample of k units, and (2) Kruskal's H test [ibid. **23**, 525–540 (1952); these Rev. **14**, 391] or Rijkoort's extension [Nederl. Akad. Wetensch. Proc. Ser. A. **55**=Indagationes Math. **14**, 394–404 (1952); these Rev. **14**, 391] of Wilcoxon's test to k samples, for the difference between k samples of m units in which all the mk units are set out in ascending order. The best approximations previously found to the theoretical distributions of uncorrelated samples are incomplete Beta functions (=the Z approximation). They are used here to obtain much simpler approximations useful for testing a significance quickly; from these, nomograms have been constructed for all significance levels between 1 and 10 percent." (From the author's summary.) *L. A. Aroian.*

Latscha, R. Tests of significance in a 2×2 contingency table: Extension of Finney's table. Biometrika **40**, 74–86 (1953).

Finney [Biometrika **35**, 145–156 (1948); these Rev. **10**, 52] has given tables to test the significance of the deviation from proportionality in any 2×2 contingency table having both the frequencies in one of its margins less than or equal to 15. The author extends Finney's tables up to and including marginal frequencies of 20 for levels of significance of .05, .025, .01, and .005. *L. A. Aroian* (Culver City, Calif.).

Henderson, C. R. Estimation of variance and covariance components. Biometrics 9, 226–252 (1953).

Fifty-seven butterfat records were obtained from four herds (h) using three sires (s) over a period of four years (a). The linear model representing the yearly production of butterfat (Y) was $Y_{hijk}=\mu+a_h+h_i+s_j+(hs)_{ij}+e_{hijk}$, where e represents a random error and (hs) the herd-sire interaction. The numbers of records differed widely for various herd-sire-year combinations, with many vacant cells. Three computing procedures are presented to estimate the parameters in this model, under different assumptions: (1) All effects are random, except μ, with respective variances σ_a^2, σ_h^2, σ_s^2, σ^2_{hs}, σ_e^2. Only these variance components and μ need be estimated. (2) Same as (1) except the a_h are fixed and must be estimated. Two computing schemes are presented: (a) Estimate μ and the a_h by least squares, adjust the data for these effects and estimate the variance components as in (1) after taking account of the adjusting procedures. (b) Compute mean squares by least squares, regarding all effects but e to be fixed, find expectations of corresponding mean squares in terms of the variance components and solve simultaneously for the estimates of these components.

Model (2) is often the more realistic and computing method (b) corrects for possible correlation biases; however, (b) usually will involve a prohibitive amount of computing. This reviewer has developed a similar tedious computing procedure when two components are fixed (plus the mean). Unfortunately, the efficiency of the estimates of the variance components may be quite low for all of these computing schemes for disproportionate data which are based on obtaining unbiased estimates from the analysis of variance. More information is needed on the relative efficiencies of estimates based on the different computing procedures, and on the possibility of developing more efficient estimates (possibly biased ones). *R. L. Anderson.*

Cox, D. R., and Smith, W. L. The superposition of several strictly periodic sequences of events. Biometrika 40, 1–11 (1953).

Suppose that there are N sources, that on the ith source events occur at times $\theta_i, 2\theta_i, \cdots$, and that the outputs of the N sources are combined into a single pooled output. Statistical properties of the pooled output are investigated. Methods are suggested for distinguishing it from a random series and for estimating N from experimental data. Applications are indicated to experimental psychology and to neuro-physiology. (Authors' summary.) *L. J. Savage.*

Middleton, David. Statistical criteria for the detection of pulsed carriers in noise. I. J. Appl. Phys. 24, 371–378 (1953).

Part I describes a theory for the detection of a pulsed carrier wave in normal random noise. The detection of signals in the presence of noise is a central problem of communication, which is basically statistical. As such, to determine the presence or absence of a signal in noise is equivalent, in statistical language, to testing the hypothesis (H_0) that noise alone is present against the alternative hypothesis (H_1) of a signal and noise. A decision based on the finite amount of information at our disposal is then subject to two types of error: a type I error, of calling noise a signal when really noise alone is present, and a type II error, of considering a signal as noise when a signal is actually present; the respective probabilities of these errors we denote by α and β. In reaching a decision, the observer in effect makes a "bet", which in the limit of an infinitely large number of repeated trials has a definite probability of "failure", i.e., a faulty decision. The process of observation is then completely specified analytically by the way in which the observer uses his data to make his bet. The probability of a success depends on the ratio a_0 of the input signal strength to a (fixed) amount of interfering noise.

The relation between the percentage of successes (i.e., correct decisions) and the input signal-to-noise ratio a_0 is expressed in terms of a betting curve introduced by Lawson and Uhlenbeck [Threshold signals, McGraw-Hill, New York, 1950, Chapters 7, 8, Secs. 8.3, 8.6, 8.8] and first calculated by Siegert [loc. cit., preface]. The minimum detectable signal is defined as that signal observed in the output of the receiver for which there is some arbitrarily selected percentage of success. If p and q are the a priori probabilities of signal and noise and noise alone, the betting curve $W(a_0^2; n)$ is given by $W(a_0^2; n)=p(1-\beta)+q(1-\alpha)$, where n is the sample size and is proportional to the integration time; the probabilities α and β of the type I and II errors are in general functions of a_0^2 and n. Three types of observer are next considered; the Neyman-Pearson, the Ideal, and the Sequential observer, whose properties are determined by the manner in which the test is carried out.

S. Ikehara (Tokyo).

Middleton, David. Statistical criteria for the detection of pulsed carriers in noise. II. J. Appl. Phys. 24, 379–391 (1953).

Part II determines explicitly the betting curves for the Neyman-Pearson, Ideal, and Sequential observers, and compares the performance of these observers for the weak-signal cases, and for some approximate curves for strong signals. The input signal-to-noise (power) ratio $(a_0^2)_{\min}$ corresponding to the minimum detectable signal at the output (with an arbitrary probability of a correct decision) is found, as expected, to vary as n^{-1} for weak signals and as $n^{-\frac{1}{2}}$ for strong, where n=number of pulses integrated. Furthermore, comparison of observers on the basis of the minimum detectable signal reveals the Neyman-Pearson to be better than the Ideal, by an amount 0 (0.5 db). On the basis of the assumed approximate distribution of sample size, the Neyman-Pearson also appears superior to the Sequential observer, but the latter must always be as good as, if not better than, the former.

An analysis is given of performance and some of the factors affecting it. The paper concludes with a brief account of a statistical theory of the observation process and its relation to the various observers considered here.

S. Ikehara (Tokyo).

Goodman, Leo A. Population growth of the sexes. Biometrics 9, 212–225 (1953).

The methods of studying population growth used by D. G. Kendall [J. Roy. Statist. Soc. Ser. B. 11, 230–264 (1949); these Rev. 11, 672] are here used to study the growth of the different sexes within the population. The models (both deterministic and stochastic models are considered) are constructed to allow for different birth and death rates as between males and females. In the stochastic case, the first and second moments of the total male and female population sizes are determined in two simple models.

D. G. Chapman (Seattle, Wash.).

TOPOLOGY

***Bourbaki, N. Eléments de mathématique. XVI. Première partie: Les structures fondamentales de l'analyse. Livre III: Topologie générale (fascicule de résultats).** Actualités Sci. Ind., no. 1196. Hermann & Cie, Paris, 1953. 95 pp. 1500 francs.

The publication of Livre III, Topologie générale, of the treatise by Bourbaki has been complete for some time now. The author's purpose in now publishing this "fascicule de résultats" is explained in the introduction as follows:

"Le présent fascicule est conçu avant tout comme un répertoire: pour chacune des notions fondamentales de la Topologie, on a cherché à grouper, d'une façon aussi concise que possible, les résultats essentiels où intervient cette notion, et qu'il est bon d'avoir présents à l'esprit lorsqu'on doit l'utiliser. En particulier, on a cherché à rassembler les critères qui permettent d'affirmer qu'un ensemble ou une fonction possède telle ou telle propriété, par exemple être compact ou être continue. Ce mode d'exposition entraîne d'une part des redites: par exemple, le théorème affirmant que l'image d'un ensemble compact par une application continue (dans un espace séparé) est compacte, se trouve à la fois dans le paragraphe consacré aux fonctions continues et dans celui consacré aux espaces compacts. D'autre part, il arrive très fréquemment qu'une notion soit mentionée dans un paragraphe antérieur à celui où elle est définie: L'index terminologique placé à la fin de ce fascicule renvoie alors à l'endroit exact du fascicule où est donnée la définition. . . . La théorie des nombres réels est supposée connue du lecteur.

"Conformément à l'esprit de ce fascicule, on ne trouvera aucune démonstration des résultats énoncés; pour les théorèmes les plus difficiles, un renvoi indique à quel endroit du Livre III se trouve la démonstration."

It should be emphasized that this volume summarizes all the definitions, terminology, and results of Livre III; no proofs, examples, or exercises are given.

W. S. Massey (Providence, R. I.).

Papić, Pavle. Sur une classe d'espaces abstraits. C. R. Acad. Sci. Paris **236**, 1843–1845 (1953).

L'auteur appelle espaces R les espaces accessibles (au sens de M. Fréchet) admettant une base ramifiée O de voisinages. Ces espaces avaient été antérieurement considérés par G. Kurepa [Publ. Math. Univ. Belgrade **5**, 124–132 (1936)]. Une base ramifiée O de voisinages est une famille d'ensembles ouverts non vides de l'espace telle que tout ensemble ouvert non vide de l'espace est réunion de certains ensembles de O, et que: pour tous $O' \in O$ et $O'' \in O$, ou bien $O'' \subset O'$, ou bien $O' \subset O''$, ou bien O' et O'' sont disjoints. Par exemple, l'espace à zéro dimension de Baire est un space R. L'auteur énonce de nombreuses propriétés des espaces R, en particulier des propriétés en relation avec la théorie des tableaux ramifiés de G. Kurepa et les espaces à écart abstrait ordonné de Kurepa-Fréchet. *A. Appert.*

Colmez, Jean. Sur les espaces précompacts. C. R. Acad. Sci. Paris **233**, 1552–1553 (1951).

R. Doss [Ann. of Math. (2) **48**, 843–844 (1947); these Rev. **9**, 243] proved that a uniformisable T_0-space admits only precompact structures if and only if its image by every continuous real-valued function is compact (actually the result given in the above-mentioned paper is slightly different from, but is equivalent to, the above statement). The author calls such spaces precompact. He then states, among others, the following results. If E is completely regular, then these propositions are equivalent: (1) E admits only precompact structures; (2) every continuous real image of E is bounded; (3) every continuous metric image of E is compact; (4) from every denumerable open covering of E, we can extract a finite number of open sets such that the union of their adherences contains E; (5) every filter with an open denumerable base has an adherent point. ((1)$\leftrightarrow$(2) is essentially the result of Doss, loc. cit.) If the number of precompact structures that a completely regular space admits is inferior to $2^{\aleph_0}$, then it is precompact. Let E be a precompact space; if a belongs to a completion of E but a non-$\in E$, then every denumerable family of neighborhoods of a has a non-void intersection with E. Every normal precompact space is semi-compact. The adherence of any open set in a precompact space is precompact. A paracompact precompact space is compact.

A completely regular space E is called uniformly locally precompact if there exists a vicinity V, such that $V(a)$ is precompact for every $a \in E$. Let A be a precompact set in a uniformly locally precompact space; then there exists a real-valued function f continuous over E such that $0 \leqq f(x) \leqq 1$ for $x \in E$, $f(x)=1$ for $x \in A$, and the support of f (i.e., $\overline{[x \mid f(x) \neq 0]}$) is precompact. In a uniformly locally precompact space, the vector space $\mathcal{S}$ of the real-valued continuous functions with precompact supports has the following properties: (1) it is a lattice; (2) if $f \in \mathcal{S}$, then $\inf(f, 1) \in \mathcal{S}$; (3) if $I(f)$ is a positive linear functional over $\mathcal{S}$, and if the monotonic sequence $f_1, f_2, \cdots, f_n, \cdots$ has $f \in \mathcal{S}$ as limit, then $I(f_n) \to I(f)$. In these spaces, a theory of integration analogous to that for locally compact spaces can be developed; the rôle played by semi-continuous functions being taken over by superior and inferior bounds of denumerable families of continuous functions. *H. Tong.*

Colmez, Jean. Sur les espaces précompacts. C. R. Acad. Sci. Paris **234**, 1019–1021 (1952).

The author continues his work on precompact spaces [see the paper reviewed above]. Further interesting results are announced; among them are the following. A locally precompact space is a Baire space. Every real-valued upper semi-continuous function on a precompact space which is the limit of a sequence of continuous functions assumes its maximum. If E is uniformly locally precompact, then its completion in any uniform structure possessing a vicinity V such that $V(x)$ is precompact for every $x \in E$ is locally compact. If F is a metric image of a uniformly locally precompact space by a certain continuous function f (explained in this note), then every measure defined by the procedure indicated in the previous note induces in F a Radon measure defined by putting $I(g)=I(g(f))$, where g is a real-valued continuous function over F with a compact support. He also defines (and gives results about) a Haar measure on locally precompact groups (such groups being uniformly locally precompact). The note ends with a theory of integration in locally precompact spaces without uniformity.

H. Tong (New York, N. Y.).

Efremovič, V. A. The geometry of proximity. I. Mat. Sbornik N.S. **31(73)**, 189–200 (1952). (Russian)

A proximity ("infinitesimal") space is, in the most general case, simply a set R with a binary relation δ defined for

subsets (example: if R is a metric space, let $A \delta B$ if and only if dist $(A, B) = 0$). The following axioms are introduced: (0) $A \delta B$ implies $B \delta A$; (1) $(A \cup B)$ non-δ C if and only if A non-δ C, B non-δ C; (2) $a \delta b$ if and only if $a = b$ (a, b are points of R); (3) if A non-δ B, then there exist C, D such that $C \cap D = 0$, A non-δ $(R - C)$, B non-δ $(R - D)$.

Main results: if A non-δ B, then there exists a uniformly continuous (i.e., such that $f(A) \delta f(B)$ whenever $A \delta B$) real-valued function f with $f(A) = (0)$, $f(B) = (1)$; given a compact topological space R, there is only one proximity relation δ compatible with R (i.e., such that $x \in \bar{M}$ if and only if $x \delta M$). *M. Katětov* (Prague).

Smirnov, Yu. On proximity spaces in the sense of V. A. Efremovič. Doklady Akad. Nauk SSSR (N.S.) **84**, 895–898 (1952). (Russian)

The author considers proximity spaces [cf., e.g., the paper reviewed above] and their relations with uniform spaces. Many results are given without proofs.

A proximity space (briefly, δ-space) R is called maximal if it is closed in any δ-space $S \supset R$. It is shown that a δ-space is maximal if and only if it is compact (as a topological space); every δ-space may be imbedded (in an essentially unique way) in a maximal δ-space; if R is a given completely regular topological space, then there is a one-to-one correspondence between proximity structures on R compatible with its topology and compact spaces containing R topologically as a dense subset.

For a given δ-space R consider the collection (partially ordered in an obvious way) of all uniformities compatible with its proximity structure. It is stated that this collection has a minimum; if R is metrizable, it has a maximum. If such a maximal uniformity exists but does not admit of an extension onto a δ-space $S \supset R$, $S \neq R$, $S = \bar{R}$, then R is called complete. It is stated that any δ-space for which there exists a maximal uniformity may be imbedded in a complete δ-space.

It is to be noted that Theorem 11 of the article is not correct (as pointed out by the author in another note [same Doklady (N.S.) **88**, 761–764 (1953), last footnote on p. 762]). *M. Katětov* (Prague).

Smirnov, Yu. M. On proximity spaces. Mat. Sbornik N.S. **31** (**73**), 543–574 (1952). (Russian)

The note contains detailed proofs of most theorems of the author's paper reviewed above. *M. Katětov.*

Dickinson, Alice. Compactness conditions and uniform structures. Amer. J. Math. **75**, 224–228 (1953).

The principal results of this paper are as follows. The coarsest uniform structure (compatible with the topology) of a locally compact (Hausdorff) space is that induced by its (at most) one-point compactification. A space with a unique uniform structure has a unique compactification, and is both locally compact and countably compact. Finally, a paracompact space is complete with respect to its universal uniform structure; this answers a question raised by Dieudonné [J. Math. Pures Appl. (9) **23**, 65–76 (1944); these Rev. **7**, 134]. *A. H. Stone* (Manchester).

Mazur, S. On continuous mappings on Cartesian products. Fund. Math. **39** (1952), 229–238 (1953).

Let ϕ be a sequentially continuous mapping of a space A in a space B. This paper gives conditions on A and B under which ϕ will in fact be continuous. For example, it suffices that B be metrizable and that A be the Cartesian product of $\mathfrak{m}$ Hausdorff spaces with countable bases, where $\mathfrak{m}$ is less than the first inaccessible aleph. The method is to show that (under these hypotheses) ϕ must have the form $\phi_0 \pi$, where π is the projection of A onto the product A_0 of only countably many of the factors, and where ϕ_0 is sequentially continuous, and therefore continuous, on A_0. This is proved by means of a theorem on infinite cardinals which generalizes Ulam's theorem on measures [Fund. Math. **16**, 140–150 (1930)]. [The statement about $\mathfrak{G}_\delta$-classes on p. 231, lines 8–10 seems to the reviewer to be false, and the definition of $\mathfrak{B}$-class (p. 231, condition (a)) should be slightly modified accordingly.] *A. H. Stone* (Manchester).

Henriksen, Melvin, and Isbell, J. R. On the continuity of the real roots of an algebraic equation. Proc. Amer. Math. Soc. **4**, 431–434 (1953).

Let X be a Hausdorff space, $C(X, R)$ the ring of real-valued continuous functions on X, M a maximal ideal of this ring, and C_M the corresponding residue class ring $C(X, R)/M$. The authors show that, if X is normal, C_M is a real closed field (i.e., it is ordered, every positive element of C_M has a square root in C_M, and every polynomial of odd degree with coefficients in C_M has a root in C_M). This result was stated by E. Hewitt for the case that X is completely regular, but part of his proof depended on the argument that the least real root of a polynomial of odd degree depends continuously on the coefficients of the polynomial [Trans. Amer. Math. Soc. **64**, 45–99 (1948); these Rev. **10**, 126]. That this argument is invalid was noted by J. Dieudonné in his review of Hewitt's paper [these Rev. **10**, 126]. The authors of the present paper prove the following theorem: Let X be a Hausdorff space, $\alpha = (a_0, \cdots, a_{n-1})$ an $n$tuple of complex numbers, and $P(\alpha; w) = w^n + \sum_{j=0}^{n-1} a_j w^j$. Let $\alpha = \alpha(x)$ be a continuous function on X and suppose $P(\alpha(x); w) = 0$ has a real root for each $x \in X$. Then there exists an open set U in X and a continuous real-valued function $r(x)$ on U such that $P(\alpha(x); r(x)) = 0$ if $x \in U$. If the number of real roots of $P(\alpha(x); w) = 0$ is the same for each $x \in X$, all of X may be taken for U. With this theorem and Tietze's extension theorem the authors prove that C_M is a real closed field. The normality of X enters in using Tietze's theorem.

A. E. Taylor (Los Angeles, Calif.).

Horn, Alfred. The normal completion of a subset of a complete lattice and lattices of continuous functions. Pacific J. Math. **3**, 137–152 (1953).

The author gives a direct construction of the normal completion of a partially ordered subset of a complete lattice, and uses this construction to extend results of Dilworth [Trans. Amer. Math. Soc. **68**, 427–438 (1950); these Rev. **11**, 647] concerning function-lattices and set-lattices connected with a topological space. Let C be a subset of a complete lattice B. If $x \in B$, let

$$(x_0) = \{y \in C \mid y \leqq x\}, \quad (x^0) = \{y \in C \mid y \geqq x\}, \quad x_* = \sup (x_0),$$
$$x^* = \inf (x^0), \quad L = \{x \in B \mid x = x_*\}, \quad U = \{x \in B \mid x = x^*\},$$
$$NL = \{x \in B \mid x = (x^*)_*\}, \quad NU = \{x \in B \mid x = (x_*)^*\}.$$

Then the author proves that NL and NU are isomorphic to the normal completion $\bar{C}$ of C. In particular, this applies to the lattice B of all real functions on a topological space S with ∞ and $-\infty$ allowed as values, and the subset C of B consisting of all real continuous functions on S. If S is completely regular, $\bar{C}$ is isomorphic to the lattice of all normal lower semi-continuous functions on M; furthermore,

if C_1 denotes the set of all finite real continuous functions on S, and C_2 the set of all bounded real continuous functions on S, then $\bar{C}_1$ is isomorphic to the lattice consisting of all normal lower semi-continuous functions on S which are bounded above and below by members of C_1, together with the functions which are identically ∞, $-\infty$, while $\bar{C}_2$ is isomorphic to the lattice of all bounded normal lower semi-continuous functions, together with ∞, $-\infty$. The lattices of regular open sets of two topological spaces S_1 and S_2 are isomorphic if and only if the lattices of normal lower semi-continuous functions are isomorphic. *S. B. Myers.*

Arens, Richard. Extension of coverings, of pseudometrics, and of linear-space-valued mappings. Canadian J. Math. **5**, 211–215 (1953).

Let A be a closed subset of a topological space X. The author proves the equivalence of three statements about A and X; the most significant special cases are as follows. I. The following statements are all equivalent: (a) every locally finite open covering of A can be extended to one of X; (b) every pseudometric on A can be extended to one on X; (c) every mapping of A in a closed convex subset S of a Banach space can be extended to X. II. These statements remain equivalent if we require the coverings to be countable, the pseudometrics to admit countable "dense" sets, and the set S to be separable. He gives an extension of (c) to certain non-closed sets, and notes that the statements in II are implied by the normality of X, but leaves open the corresponding question for I. [It has been answered by Dowker [Ark. Mat. **2**, 307–313 (1952); these Rev. **14**, 396], who, besides implicitly proving the equivalence of I (a) and (c), has shown that they hold for a given X and all A if and only if X is collectionwise normal.] *A. H. Stone.*

Sikorski, R. Generalized limits and means. Ann. Soc. Polon. Math. **25** (1952), 106–109 (1953).

A semigroup G is called solvable if there is a sequence $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ of subsemigroups such that for each i, $1 \leqq i \leqq n$, (a) if $\varphi \in G_i$ and $\psi \in G_{i-1}$, then there is a $\psi^* \in G_{i-1}$ such that $\psi^*\varphi = \varphi\psi$: (b) if $\varphi, \psi \in G_i$, then there is $\psi^* \in G_{i-1}$ such that $\varphi\psi = \psi^*\psi\varphi$; (c) G_0 contains only the unit element of G. This paper extends the results of the author's earlier note [Studia Math. **12**, 117–124 (1951); these Rev. **13**, 216] on the existence of average values for maps from a set T into compact convex subsets of linear topological spaces with some invariance properties under a semigroup G of functions from T into T, by weakening the hypothesis from G abelian to G solvable. *M. M. Day* (Urbana, Ill.).

Wallace, A. D. Boolean rings and cohomology. Proc. Amer. Math. Soc. **4**, 475 (1953).

Let $B(S)$ be the Boolean ring of a Stone space S, and let $\{S_\lambda\}$ be an inverse system of Stone spaces. A brief proof is given of the fact that $B(\text{inv lim } S_\lambda) \approx \text{dir lim } B(S_\lambda)$ by first proving this relation with B replaced by H^0. References are to E. H. Spanier [Ann. of Math. (2) **49**, 407–427 (1948); these Rev. **9**, 523] and to the reviewer [Proc. Amer. Math. Soc. **2**, 566–576 (1951); these Rev. **13**, 524]. *F. Haimo.*

Grzegorczyk, A., and Kuratowski, C. On Janiszewski's property of topological spaces. Ann. Soc. Polon. Math. **25** (1952), 169–182 (1953).

Let S be a connected, locally connected, normal T_1 space, and suppose that, whenever A and B are closed connected subsets of S such that $S-(A \cup B)$ is connected, then $A \cap B$ is connected. The authors deduce that, whenever C, D are components of any closed subsets A, B of S such that $S-(A \cup B)$ is connected, then $C \cap D$ is connected; and that this in turn implies the same property with "closed" replaced by "open". If further S is compact metric, all these properties are known to be equivalent to one another [from the fact that they make every true cyclic element of S a 2-sphere; see C. Kuratowski, Fund. Math. **13**, 307–318 (1929)]. In the general case, the question of equivalence remains open. In an appendix it is pointed out that the results extend to a suitable class of "closure algebras" [R. Sikorski, Fund. Math. **36**, 165–206 (1949); these Rev. **12**, 85]. *A. H. Stone* (Manchester).

***Katětov, Miroslav. On the dimension of nonseparable metric spaces.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 359–362. Akadémiai Kiadó, Budapest, 1952. (Russian and Hungarian)

A detailed version of this paper appeared in Časopis Pěst. Mat. Fys. **75**, 79–87 (1950) [these Rev. **12**, 119].

Sierpiński, Wacław. Sur une homéomorphie de classe 1, 1 entre un segment de droite et un carré. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **44** (1951), 62–65 (1952).

Let S denote the segment $-1 \leqq t \leqq 1$, and Q denote the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. The author [Revista Mat. Hisp.-Amer. (2) **2**, 193–197 (1927)] has given a one-to-one correspondence T between S and Q, defined by $x=\phi(t)$, $y=\psi(t)$, where ϕ and ψ are continuous on the left at every point of S. A consequence of a result proved by Kuratowski [Fund. Math. **22**, 206–220 (1934), p. 212] is that there exists a correspondence between S and Q which is a homeomorphism of class 1, 1. In the note under review, the author shows that T is such a correspondence. *F. Bagemihl.*

Thimm, Walter. Untersuchungen über Deformationen. Math. Ann. **125**, 19–31 (1952).

In a metric space R let $A(t)$, $0 \leqq t \leqq 1$, be a family of closed sets that has the following weak continuity property: to every ϵ there is a δ such that $A(t)$ is contained in the ϵ-neighborhood of $A(t_0)$ whenever $t_0 \leqq t \leqq t_0+\delta$. Suppose furthermore that, for each t, there is a deformation ρ_{tr}, $0 \leqq \tau \leqq 1$, of $R-A(t)$ over itself such that (1) ρ_{tr} leaves fixed the points outside the τ-neighborhood of $A(t)$, and (2) $A(t)$ is disjoint to $\rho_{tr}(R-A(t_0))$ whenever $t_0 \leqq t \leqq t_0+\delta$. Suppose given a map c of the set

$$\{t=0, 0 \leqq \tau \leqq 1\} \cup \{0 \leqq t \leqq 1, \tau=0\} \cup \{0 \leqq t \leqq 1, \tau=1\}$$

such that $c_0(\tau) \subset R-A(0)$, and $c_t(0) \cup c_t(1)$ lies outside the ϵ-neighborhood of $A(t_0)$ whenever $|t-t_0|<\delta$. Then c can be extended to a map of the rectangle $\{0 \leqq t \leqq 1, 0 \leqq \tau \leqq 1\}$ in such a way that $c_t(\tau) \subset R-A(t)$.

The application that the author has in mind for this theorem is the case where the sets $A(t)$ are the branch manifolds of an algebroid function undergoing parameter specialization. In fact the theorem embodies methods used by van Kampen and Zariski to study the fundamental group of the complement in a projective space of an algebraic variety.

The rest of the paper seems to consider the possibility of finding neighborhoods of a closed set A whose complements are deformation retracts of $R-A$. The author's statement [Aufgabe III, p. 25] of the problem is apparently incomplete because, as it stands, it has a trivial solution under no

hypotheses. It is not clear to the reviewer what the problem is supposed to be, but the main result would seem to be that it has a solution if A is an algebraic variety.

There are a number of other theorems of the same general nature; they are mostly too complicated to state here.

R. H. Fox (Princeton, N. J.).

Kuratowski, C., et Zarankiewicz, C. Sur un problème concernant les coupures des régions par des continus. Fund. Math. **39** (1952), 15–24 (1953).

Given k disjoint continua $K_1, K_2, \cdots, K_k$ and n disjoint regions $R_1, R_2, \cdots, R_n$ in the 2-sphere S_2 such that no R_j separates S_2 and, for each pair i, j, $K_i \cdot R_j \neq 0$, the authors consider the problem of determining the minimum number $s_{k,n}$ of pairs K_i, R_j such that K_i separates R_j. It has been conjectured by one of the authors that $s_{k,n} = (k-2)(n-2)$ for $k \geqq 2$ and $n \geqq 2$ and in this paper this relation is established in all cases for which either $k \leqq 4$ or $n \leqq 4$. An example is given showing that this minimum for $s_{k,n}$ is possible of realization for any k and n. *G. T. Whyburn.*

Reifenberg, E. R. A separation theorem for finite sets of plane continua. Proc. Cambridge Philos. Soc. **49**, 573 (1953).

It is shown that if a finite union of plane continua with a common point separates a pair of points, then the union of some two of the continua also separates the points.

A. D. Wallace (New Orleans, La.).

Hamstrom, Mary-Elizabeth. Concerning webs in the plane. Trans. Amer. Math. Soc. **74**, 500–513 (1953).

Let M be a (compact) continuum in the plane, and let n be an integer >1. Suppose that there is a collection F, of n elements, such that (1) each element of F is an upper semi-continuous collection of disjoint continua which fills up M and is an arc with respect to its elements, and (2) if G is a collection of continua each belonging to some, but no two to the same, element of F, then the elements of G have a common point and their intersection is totally disconnected. Then M is a W_n set. If there is an F, as above, such that each element of F is a continuous collection, then F is a W_n' set. (For example, the decompositions of the unit square C into horizontal intervals and vertical intervals show that C is (trivially) a W_2' set.) It is shown that there is a W_2 set which is not a W_3 set, and that a 2-cell is a W_n set for each n. Also, it is shown that (plane) W_3' sets are topologically characterized by certain conditions imposed on their boundaries, expressed in terms of the "contiguous-point" spaces of R. L. Moore [Rice Inst. Pamphlet **23**, 1–74 (1936), pp. 43 and 56]. The following are typical of the further results. (I) Let M be an annulus (=closed region bounded by two concentric circles). Then there is a collection F of n elements, such that each element of F is a collection of arcs, and such that F satisfies the definition of a W_n' set, except that the elements of F are simple closed curves with respect to their elements. (II) The limiting set of any infinite sequence of boundaries of different components of the complement of a W_3 set is totally disconnected. (III) Every annulus is a W_n set for each n. *E. E. Moise.*

Deheuvels, René. Points critiques d'une fonctionnelle. C. R. Acad. Sci. Paris **236**, 1847–1849 (1953).

This is a continuation of the author's previous work [same C. R. **235**, 778–780, 858–860, 929–930, 1270–1272 (1952); these Rev. **14**, 492]. The author first defines the notion of critical points of a numerical function and then announces the following: Let p be a critical value of degree n of a numerical function φ on X. If the sets $[\varphi \leqq p+\epsilon]$ are compact for $0<\epsilon<\epsilon_0$, there exists at least a critical point of φ for the value p, of index $\leqq n$. *C. T. Yang.*

Krasnosel'skiĭ, M. A. On the theory of completely continuous vector fields. Ukrain. Mat. Žurnal **3**, 174–183 (1951). (Russian)

Let F be a completely continuous operator defined on the sphere S, $\|x\| = \rho$, of some Banach space E, and let Φ be the vector field $Fx - x$, $x \in S$, which is assumed to contain no zero vector. Then the characteristic $\chi(\Phi, S)$ of Φ on S is defined, generalizing the definition used for vector fields on finite-dimensional spheres. The author proves that two such vector fields Φ, Ψ are homotopic if and only if $\chi(\Phi, S) = \chi(\Psi, S)$. He mentions that this was proved by E. Rothe for the special case where E is a Hilbert space, but the latter actually proved it for the general case of a vector field defined on the boundary of a convex set in a strictly convex linear topological space [Iowa State Coll. J. Sci. **13**, 373–390 (1939); these Rev. **1**, 108]. In the second part of the paper the author extends a theorem proved by Borsuk and Lusternik-Schnirelmann for the finite-dimensional case to E: If the vectors of the field Φ are not parallel at diametrically opposite points of S then $\chi(\Phi, S)$ is odd. It follows that under these conditions if F is defined and completely continuous for $\|x\| \leqq \rho$, F has a fixed point in the interior of S. With this fixed-point principle the author proves that if A is defined and completely continuous for $\|x\| \leqq \rho$, and B is a linear completely continuous operator such that $\|Ax - \lambda Bx\| < \|x - \lambda Bx\|$ for $x \in S$ then $Ax = x$ has a solution x_0, $\|x_0\| < \rho$. It seems to the reviewer that the conditions on B are unnecessarily restrictive. The same result follows if $\|Ax + Bx\| < \|x + Bx\|$, $x \in S$, for any operator B for which $B(-x) = -Bx$, $x \in S$. *M. Golomb* (Lafayette, Ind.).

Fan, Ky. Minimax theorems. Proc. Nat. Acad. Sci. U. S. A. **39**, 42–47 (1953).

The author gives a formulation of von Neumann's minimax theorem for some spaces with no linear structure. [For material in the same spirit see the remarks in the review of K. Fan's previous paper, same Proc. **38**, 121–126 (1952); these Rev. **13**, 858; and the paper by G. Debreu, ibid. **38**, 886–893 (1952); these Rev. **14**, 301.] *D. G. Bourgin.*

***Stoilow, Simon. Sur les transformations intérieures des variétés à trois dimensions.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 263–266. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

Given a light open mapping f of one 3-dimensional manifold X into another one Y, the author announces the result that the dimensionality of the set of points of X where f is not locally topological is $\leqq 1$. The proof which is outlined effects a reduction to the case of a light open mapping on a 2-dimensional manifold by using a theorem of R. L. Wilder giving conditions under which the common boundary of two regions in E_2 is a 2-manifold. Known results in the latter case together with a theorem of Alexandroff on open mappings of countable multiplicity can then be applied. The paper concludes with the statements of two related problems which are as yet unsolved. *G. T. Whyburn.*

Heller, Alex. On equivariant maps of spaces with operators. Ann. of Math. (2) **55**, 223–231 (1952).

Let $G=\{g\}$ be a group operating on two topological spaces X and Y, and hence on the cartesian product $X\times Y$ of X and Y by $g(x, y)=(gx, gy)$. A continuous map f of X into Y is said to be G-equivariant if $fg=gf$ for all $g\in G$. The author first obtains necessary conditions for the existence of a G-equivariant map of X into Y. It is necessary that, for any homology theory H and for any dimension q, the group $H_q(X/G)$ be isomorphic to a direct summand of the group $H_q((X\times Y)/G)$, where X/G and $(X\times Y)/G$ are decomposition spaces whose points are orbits of G in X and $X\times Y$, respectively. It is also necessary that $H_q(X_0/\Gamma)$ be isomorphic to a direct summand of $H_q((X_0\times Y)/\Gamma)$ for any subgroup Γ of G and for any Γ-invariant subset X_0 of X.

These results are then applied to the case when $X=R_{n+1}=$ the group of all orientation-preserving rotations of an $(n+1)$-space R^{n+1} with an orthonormal system $(e_0, \cdots, e_n)$, $Y=S^{n-1}=$ the $(n-1)$-sphere consisting of all unit vectors of R^{n+1} which are orthogonal to the vector $e_0+\cdots+e_n$, and $G=A_{n+1}=$ the subgroup of R_{n+1} consisting of all rotations of R^{n+1} which permute $e_0, \cdots, e_n$ (even permutation!), and hence leave invariant the vector $e_0+\cdots+e_n$. (A_{n+1} operates on S^{n-1} in an obvious way, while A_{n+1} operates on R_{n+1} by the formula: $\alpha\circ\rho=\rho\alpha^{-1}$, $\alpha\in A_{n+1}$, $\rho\in R_{n+1}$.) It is shown that there is no A_{n+1}-equivariant map of R_{n+1} into S^{n-1} if $n=2$ or $n=3$. (The problem is still open if $n>3$.) If $n=2$, this follows from the fact that (denoting by I_p the cyclic group of order p), with coefficients I_2, $H_2((R_3\times S^1)/A_3)=0$, $H_2(R_3/A_3)\approx I_3$. If $n=3$, this follows from the fact that, with coefficients I_4, $H_1((\bar{R}_3\times S^2)/\Gamma)\approx I_4\times I_2$, $H_1(R_3/\Gamma)\approx I_2\times I_2$, where $\bar{R}_3$ is the subset (in fact, subgroup) of R_4 consisting of all rotations of R^4 which leave invariant the vector $e_0+\cdots+e_3$ and Γ is the subgroup of A_4 generated by the transpositions (e_0, e_1) and (e_2, e_3).

From these results follows the assertion of the frame problem for $n=2$ and $n=3$; for any real-valued continuous function f defined on the n-sphere S^n, there exists an orthonormal system $(e_0, \cdots, e_n)$ from S^n, such that $f(e_0)=\cdots=f(e_n)$. [The frame problem was solved for general n by H. Yamabe and Z. Yujôbô, Osaka Math. J. **2**, 19–22 (1950); these Rev. **12**, 198.] The author also discusses the application of the notion of equivariant maps to the problems of fibre bundles and tensors on differentiable manifolds. *S. Kakutani.*

Noguchi, Hirochi. On property of manifolds in the sense of Poincaré. Mem. Fac. Sci. Eng. Waseda Univ. **16**, 118–119 (1952).

Proof of the following theorem: An n-dimensional finite complex K is a manifold if, and only if, given a pair of its points, a and b, there is a continuous family H_t of homeomorphisms of K onto itself such that H_0 is the identity and $H_1(a)=b$. *E. G. Begle* (New Haven, Conn.).

James, I. M. Note on factor spaces. J. London Math. Soc. **28**, 278–285 (1953).

This concerns the homotopy properties of factor spaces of a Lie group G. Let K, L, H be closed subgroups of G and K, L belong to H. Concerning the homotopy sequence

$$\cdots\to\pi_i(G/K)\xrightarrow{q_*}\pi_i(G/H)\xrightarrow{\Delta}\pi_{i-1}(H/K)\to\cdots$$

the following is proved: "Suppose that G/L is connected and L is conjugate to a subgroup of K in G. If H is a split extension of K by L, then, in the above sequence, q_* is an isomorphism and Δ is onto and, for $i\geq 2$,

$$\pi_i(G/H)\approx\pi_i(G/K)+\pi_{i-1}(H/K).$$"

On account of this theorem, the calculation of the homotopy groups of some factor spaces of the classical groups is reduced to that of the various Stiefel manifolds and group spaces. The author also shows, among other results, that each field of r-planes ($2r\leq n$) over S^n can be spanned by an r-field if and only if $\pi_{n-1}(O_r)=0$ where O_r denotes the orthogonal group in r-dimensional euclidean space.

H. C. Wang (Auburn, Ala.).

Whitehead, George W. On the Freudenthal theorems. Ann. of Math. (2) **57**, 209–228 (1953).

Dans ce travail l'auteur établit la suite exacte

$$(1)\quad \pi_q(S_n)\xrightarrow{E}\pi_{q+1}(S_{n+1})\xrightarrow{H'}\pi_{q-1}(S_{2n-1})\xrightarrow{P}\pi_{q-1}(S_n)\xrightarrow{E}\pi_q(S_{n+1})\to\cdots,$$

valable pour $q\leq 3n-2$; E désigne la suspension, $H'=E^{-2}H$, et $P(\alpha)=[i_n, i_n]\circ\alpha$, où i_n désigne le générateur de $\pi_n(S_n)$.

L'auteur déduit (1) de la suite exacte d'homotopie du couple (Ω_{n+1}, S_n), la sphère S_n étant convenablement plongée dans l'espace Ω_{n+1} des lacets de S_{n+1}. Utilisant le fait que $H_i(\Omega_{n+1})=Z$ (resp. 0) si $i\equiv 0 \bmod n$ (resp. si $i\not\equiv 0 \bmod n$), il montre d'abord que $\pi_q(\Omega_{n+1}, S_n)\approx\pi_{q-1}(S_{2n-1})$ pour $q\leq 3n-2$. [Note du rapporteur: en fait ceci vaut pour $q\leq 4n-3$ lorsque n est impair, ou, n étant pair, lorsqu'on "néglige" les groupes finis dont l'ordre est une puissance de 3; cela résulte de la structure multiplicative de la cohomologie de Ω_{n+1}. Cf. J.-P. Serre, C. R. Acad. Sci. Paris **234**, 1340–1342 (1952), prop. 2; ces Rev. **13**, 675.] L'auteur montre ensuite que les homomorphismes obtenus à partir de ceux de la suite exacte d'homotopie de (Ω_{n+1}, S_n) en identifiant $\pi_q(\Omega_{n+1})$ à $\pi_{q+1}(S_{n+1})$, et $\pi_q(\Omega_{n+1}, S_n)$ à $\pi_{q-1}(S_{2n-1})$, coïncident bien avec E, H' et P. Pour E, c'est immédiat; pour P, cela résulte du fait que $H_*(\Omega_{n+1})$, munie du produit de Pontrjagin, est une algèbre de polynomes; pour H', la démonstration est beaucoup plus délicate; elle s'appuie notamment sur le résultat suivant, intéressant en lui-même: si $q\leq 3n-2$, tout élément de $\pi_{q+1}(S_{n+1})$ peut être obtenu en appliquant la construction de Hopf à une application de $S_{q-n}\times S_n$ dans S_n de type (α, i_n).

Erratum: dans la suite exacte de l'Introduction (p. 209), il faut remplacer $\pi_{2n-1}(S_n)$ par $\pi_{3n-3}(S_n)$. *H. Cartan.*

Serre, Jean-Pierre. Quelques calculs de groupes d'homotopie. C. R. Acad. Sci. Paris **236**, 2475–2477 (1953).

In previous notes [same C. R. **234**, 1243–1245, 1340–1342 (1952); these Rev. **13**, 675] the author has determined the following homotopy groups of spheres: $\pi_{n+3}(S^n)$, $\pi_{n+4}(S^n)$, and $\pi_{n+5}(S^n)$ for all values of n. In this note he determines $\pi_{n+6}(S^n)$, $\pi_{n+7}(S^n)$, and $\pi_{n+8}(S^n)$ for all n. The results are as follows (Z indicates an infinite cyclic group and Z_k denotes a cyclic group of order k): $\pi_9(S^5)=Z_2$, $\pi_{10}(S^6)=Z_2+Z_{24}$, and $\pi_{n+6}(S^n)=Z_2$ for $n>4$. The groups $\pi_{n+7}(S^n)$ are as follows for $n=3, 4, \cdots, 8$: Z_{15}, Z_{15}, Z_{30}, Z_{60}, Z_{120}, $Z+Z_{120}$. For $n>8$, $\pi_{n+7}(S^n)=Z_{240}$. The groups $\pi_{n+8}(S^n)$ for $n=3, 4, \cdots, 9$ are as follows: Z_2, Z_2, Z_2, Z_2+Z_{24}, $Z_2+Z_2+Z_2$, $Z_2+Z_2+Z_2+Z_2$, $Z_2+Z_2+Z_2$, and $\pi_{n+8}(S^n)=Z_2+Z_2$ for $n>9$. The author states some preliminary lemmas leading up to the above results. Among other things, he determines the 7- and 8-dimensional homotopy groups of the orthogonal groups.

W. S. Massey (Providence, R. I.).

Nakaoka, Minoru. Exact sequences $\sum_p(K, L)$ and their applications. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 3, 83–100 (1952).

J. H. C. Whitehead [Ann. of Math. (2) 52, 51–110 (1950); these Rev. 12, 43] has described an algebraic procedure for deriving an exact sequence under certain conditions. He applied this procedure to the relative and absolute homotopy groups of the successive skeletons of a cell complex to obtain an exact sequence involving the homotopy and homology groups of the complex, and the natural homomorphism of the homotopy groups into the homology groups. In this paper, the author applies the same procedure to the Borsuk-Spanier cohomotopy groups of the various skeletons of a finite-dimensional complex K, and thus obtains an exact sequence $\Sigma_p(K)$ for each integral value of p. These exact sequences involve, among other groups, the cohomotopy groups of K, and the cohomology groups of K with coefficients in certain homotopy groups of spheres. The sequence $\Sigma_0(K)$ is a generalization of an exact sequence given by E. Spanier [ibid. 50, 203–245 (1949); these Rev. 10, 559]. All of the exact sequences $\Sigma_p(K)$ are contained in the cohomotopy exact couple of K, introduced by the reviewer in the paper reviewed below.

The author gives a couple of applications of these exact sequences. (a) He determines the structure of the n-dimensional cohomotopy group of a complex K which is $(n+2)$-dimensional and $(n-1)$-connected in terms of its cohomology groups and certain squaring operations ($n \geqq 5$). (b) He proves that the homotopy type of a polyhedron K which is $(n-1)$-connected and at most $(n+2)$-dimensional is determined by the exact sequence of Spanier referred to the above ($n \geqq 3$). *W. S. Massey* (Providence, R. I.).

Massey, W. S. Exact couples in algebraic topology. III, IV, V. Ann. of Math. (2) 57, 248–286 (1953).

Ce mémoire fait suite à un autre [Ann. of Math. (2) 56, 363–396 (1952); ces Rev. 14, 672]. La partie III étudie le "couple exact" de cohomotopie d'un complexe fini K. Soit K^p le p-squelette de K; la suite spectrale de ce couple exact peut être obtenue en associant à chaque paire d'entiers p, $p'(p>p')$ les groupes de cohomotopie relatifs $\pi^n(K^p, K^{p'})$; comme ils ne sont définis que pour $2n>p+1$, on est obligé de donner des définitions spéciales pour $2n \leqq p+1$. Le terme E_1 de la suite spectrale (notation de Serre) est défini par $E_1^{p,-q}=\pi^{p-q}(K^p, K^{p-1})$ si $p>2q+1$ (nul si $q<0$); de plus, $E_1^{2q+1,-q}$ est le noyau de $\pi^{q+2}(K^{2q+2}, K^{2q+1}) \to \pi^{q+2}(K^{2q+2})$, et $E_1^{p,-q}=0$ pour $p<2q+1$. Le terme E_2 de la suite spectrale est un invariant du type d'homotopie de l'espace K; on a $E_2^{p,-q}=H^p(K, \mathfrak{G}_q)$, $p>2q+2$, en notant $\mathfrak{G}_q$ le groupe de cohomotopie "stable" $\pi^{p-q}(S^p) \approx \pi_{n+q}(S^n)$, $n=p-q$. La différentielle d_2: $E_2^{p,-q} \to E_2^{p+2,-q-1}$, pour $p>2q+2$, s'identifie au carré de Steenrod Sq^2: $H^p(K, \mathfrak{G}_q) \to H^{p+2}(K, \mathfrak{G}_{q+1})$ relatif à une application convenable $\mathfrak{G}_q \otimes \mathfrak{G}_q \to \mathfrak{G}_{q+1}$ (résultat inédit de J. Adem). Enfin le terme E_∞ de la suite spectrale est donné, pour $2p-2q>\dim K+1$, par

$$E_\infty^{p,-q}=\pi^{p-q,p-1}(K)/\pi^{p-q,p}(K),$$

où $\pi^{n,m}(K)$ désigne le noyau de $\pi^n(K) \to \pi^n(K^m)$.

Dans la partie IV, l'auteur étudie la dualité entre le couple exact d'homotopie (partie II) et le couple exact de cohomotopie d'un complexe fini K. Elle provient d'une application $\pi_{p+q}(K^p, K^{p'}) \otimes \pi^{p+q-r}(K^p, K^{p'}) \to \mathfrak{G}_r$. L'application $E^2_{p,q} \otimes E_2^{p,q-r} \to \mathfrak{G}_r$ est explicitée, pour $0 \leqq q \leqq r$, $p>2(r-q)+2$, lorsque K est $(q+1)$-connexe, en se servant des isomorphismes $E^2_{p,q} \approx H_p(K, \mathfrak{G}_q)$, $E_2^{p,q-r} \approx H^p(K, \mathfrak{G}_{r-q})$ et de l'application naturelle $\mathfrak{G}_q \otimes \mathfrak{G}_{r-q} \to \mathfrak{G}_r$.

La partie V est consacrée à la suite spectrale (connue) d'un espace fibré. L'auteur se limite au cas où la base et la fibre sont des complexes finis. Le mémoire est suivi d'un Appendice où sont rappelées et complétées les principales propriétés des groupes de cohomotopie de Borsuk-Spanier.

Erratum: p. 251, ligne 14, au lieu de $p>2q+1$, lire $p<2q+1$.

H. Cartan (Paris).

Kudo, Tatsuji. Homological structure of fibre bundles. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 2, 101–140 (1952).

[This is a continuation and elaboration of another paper: same J. 1, 101–114 (1950); these Rev. 13, 56.] After a review of concepts of algebraic topology (e.g., the axioms), the author develops his version of Leray's spectral sequence theory (of which apparently only the 1946 version was available to him). The definitions (in a notation somewhat different from Leray's) are based on the relative homology groups $H(A_p, A_{p-1})$ of an increasing sequence $\{A_p\}$ of sets, rather than on the concept of a filtered differential group (similar to Eilenberg's axiomatic version). This makes it possible to apply the theory also to homotopy and cohomotopy (the A_p are the skeletons of a complex), thus bringing it into closer contact with Massey's theory of exact couples [see the paper reviewed above and reference there]. For cohomotopy one gets quickly the known theorems on the classification of maps of an n-complex into the n-, resp. $(n-1)$-, resp. $(n-2)$-sphere (except for small n). In place of the Steenrod square there appears the differential d_2 (although in the author's formulation the differentials of the spectral sequence are not defined explicitly; instead the cycles and boundaries are introduced directly (his $\mathfrak{H}$ and $\mathfrak{R}$)); d_2, which occurs in the computation of π^{n-2}, seems to be related to Adem's Φ [Proc. Nat. Acad. Sci. U. S. A. 38, 720–726 (1952); these Rev. 14, 306].

For fiber bundles the author establishes the formula (in Leray's notation) $E_2=H(B, H(F))$, and the other general facts (significance of $E_2^{p,0}$, etc.). The A_p are here the inverse images of the p-skeletons of the base B. Cup- and cap-products in the spectral sequence are treated; for this the axiomatic method has to be replaced by specific cellular constructions; it is assumed that the fiber is a homogeneous space of the structure group, a compact connected Lie group. Invariance under change of cell decomposition is proved. The Gysin and Wang sequences are derived. The last chapter treats the Pontryagin product. For an (orientable) bundle A, with fiber F, group G, base B, principal bundle A' the "principal map" of $A' \times F$ into A induces (in Leray's notation) a multiplication from $E_r(A')$ and $H(F)$ to $E_r(A)$ (field coefficients), which commutes with the differentials, and which for $r=2$ is the multiplication from $H(B) \times H(G)$ and $H(F)$ to $H(B) \times H(F)$, induced by the natural multiplication of $H(G)$ and $H(F)$ to $H(F)$. Various conclusions are derived with this mechanism, e.g. (in Leray's notation): If the map $p_\#$: $H(A') \to H(B)$ is onto, then the spectral sequences of A' and of A are trivial (Thm. 25); in cohomology, $E_k^{*0}(A')$ is generated by those primitive elements of $E_2^{*0}=H^*(G)$, which are cocycles for $d_2, \cdots, d_{k-1}$ (Thm. 27). (The known cohomology structure of compact Lie groups is used here.) The results are related to results of A. Borel, G. Hirsch, Leray, and the reviewer.

H. Samelson (Princeton, N. J.).

Fáry, István. Sur les anneaux spectraux de certaines classes d'applications. V. Application $X \to X/G$. Sur le terme E_2. C. R. Acad. Sci. Paris **236**, 1224–1226 (1953).

[For parts I–IV see same C. R. **235**, 686–688, 780–782, 1272–1274, 1467–1469 (1952); these Rev. **14**, 784.] Let G be a compact, connected Lie group which operates on an infinitely differentiable manifold X; the operations of G may have fixed points. Let $Y = G/X$ denote the space of trajectories, and $f: X \to Y$ the identification map. The author gives very brief indications of a method for defining various spectral sequences which give rise to relations between certain homological invariants attached to the operations of the group G. No applications of these methods are given. In the words of the author, "Nous avons préféré indiquer la méthode plutôt qu'énoncer les résultats les plus généraux que l'on peut obtenir." *W. S. Massey.*

Thom, René. Variétés différentiables cobordantes. C. R. Acad. Sci. Paris **236**, 1733–1735 (1953).

In this note, the author considers only manifolds which are compact and differentiable. Two oriented k-dimensional manifolds, W_0^k and W_1^k, are defined to be "cobounding" if there exists a compact oriented manifold with boundary, X^{k+1}, such that $\partial X^{k+1} = W_1^k - W_0^k$. Let Ω_k denote the set of equivalence classes of oriented k-manifolds thus defined. An addition is defined in Ω_k by defining the union of two manifolds to be a representative of the sum of their equivalence classes. With this definition, Ω_k is an abelian group. The zero class is the class of bounding manifolds. A multiplication is defined between equivalence classes by forming the cartesian product of representative manifolds. This makes the direct sum of the Ω_i into a graded, anti-commutative ring. By dropping all conditions of orientability in the above definitions, one defines two manifolds to be cobounding modulo two, the group N_k of equivalence classes of cobounding k-dimensional manifolds modulo 2, and the graded ring N which is the direct sum of the N_i. Every element of N is of order two. The main purpose of this note is to study the rings Ω and N. Partial results on these rings have been obtained previously by the author [Colloque de Topologie de Strasbourg, 1951, no. V, Univ. de Strasbourg, 1952; these Rev. **14**, 492] and V. A. Rohlin [Doklady Akad. Nauk SSSR (N.S.) **84**, 221–224 (1952); these Rev. **14**, 573].

For any closed subgroup G of the real orthogonal group $O(k)$, let $M(G)$ denote the space which the author has called "the complex associated with G" in a previous note [C. R. Acad. Sci. Paris **236**, 453–454 (1953); these Rev. **14**, 1005]. The author asserts that the groups Ω_k and N_k are isomorphic to the homotopy groups $\pi_{r+k}(M(SO(r)))$ and $\pi_{r+k}(M(O(r)))$ respectively for any integer $r > k+1$. Thus the main problem is to determine these homotopy groups. It is possible to partially determine these homotopy groups. This leads to the following more explicit results. (a) If all the Stiefel-Whitney characteristic numbers of a manifold vanish, then it is a bounding manifold mod two. (b) The ring N is isomorphic to an algebra of polynomials over the field of integers mod 2 having one generator for each dimension i not of the form $2^m - 1$. (c) The groups Ω_k for $k \leq 7$ are explicitly determined. (d) The groups Ω_k are finite unless k is divisible by four. (e) If all the Pontrjagin characteristic numbers of a manifold V vanish, then there exists an integer N such that $N \cdot V$ is a bounding manifold.

W. S. Massey (Providence, R. I.).

Hu, Szu-tsen. Cohomology relations in principal fiber spaces. Amer. J. Math. **75**, 60–78 (1953).

Ist X eine endlichblättrige reguläre Ueberlagerung des Raumes B, so ist der Cohomologiering $H(B)$ mit reellen Koeffizienten durch $H(X)$ und die Operationen der Decktransformationengruppe G in $H(X)$ bestimmt: $H(B)$ ist isomorph der Untergruppe von $H(X)$, welche bei G elementweise festbleibt. Dieses Ergebnis samt seinen Anwendungen auf die Bettischen Zahlen usw. [s. Eckmann, Bull. Amer. Math. Soc. **55**, 95–101 (1949); diese Rev. **10**, 559] wird in der vorliegenden Arbeit ausgedehnt auf den Fall, dass X ein kompakter metrischer Hauptfaserraum über B mit kompakter 0-dimensionaler Strukturgruppe G ist. Die Cohomologiegruppen werden durch stetige Cozyklen beschrieben, und der Beweis benützt Resultate des Verf. über die Cohomologie von Räumen mit topologischen Gruppen von Operatoren [Ann. of Math. (2) **55**, 391–419 (1952); diese Rev. **13**, 820] sowie einige auf der Integration über G beruhende Mittelbildungen von stetigen Cozyklen. Die Arbeit schliesst mit einer Anwendung auf eine lokale Eigenschaft von Wirkungsräumen lokal-kompakter Gruppen.

B. Eckmann (Zürich).

Wu, Wen-Tsun. Sur les classes caractéristiques des structures fibrées sphériques. Actualités Sci. Ind., no. 1183 = Publ. Inst. Math. Univ. Strasbourg 11, pp. 5–89, 155–156. Hermann & Cie, Paris, 1952.

The objects under consideration are sphere bundles, with structure group (1) $O(m)$, the orthogonal group in m variables, (2) $\hat{O}(m)$, the proper orthogonal group, (3) $O'(m)$, the unitary group in $m/2$ complex dimensions, (4) $O''(m)$, the quaternion-unitary (symplectic) group in $m/4$ quaternion dimensions. The main problem of the book is that of reduction, resp. extension, of the structure group of a sphere bundle from one of these groups to the other ("subordinate structures", as in the problem of almost complex structures on manifolds where one wants to reduce the structure group of the tangent bundle from $\hat{O}(m)$ to $O'(m)$). Chapter I contains a detailed study of the homology properties of the various Grassmann manifolds $R_{n,m}$ (m-planes in $(n+m)$-space), $\hat{R}_{n,m}$ (oriented planes), $C_{n,m}$ (complex $m/2$-planes in complex $(n+m)/2$-space), $Q_{n,m}$ (quaternion $m/4$-planes in quaternion $(n+m)/4$-space), following Chern, Ehresmann and Pontryagin, in terms of Schubert cycles. The homology type of the natural map between dual Grassmann manifolds (e.g., $R_{n,m}$ and $R_{m,n}$) is determined; this is of interest, e.g., for the consideration of tangent and normal bundle of an imbedded manifold. Chapter II is the central part of the book. It begins with the fact that the Grassmann manifolds are classifying spaces, i.e., every sphere bundle is induced by a map f of the base space into a suitable one of them, with isomorphic bundles corresponding to homotopic maps. This determines the characteristic ring of the bundle, namely the f^*-image of the cohomology ring of the Grassmann manifold and leads to the definition of Stiefel-Whitney classes (mod 2), Euler-Poincaré class, Pontryagin classes for cases (1), (2), Chern classes for case (3) and similar classes for case (4). The author's main contribution is the study of the behavior of the characteristic ring under extension of the group (e.g., from $\hat{O}(m)$ to $O(m)$, etc.); this is a construction of "universal" character and amounts to the study of the homology behavior of the natural maps of the respective Grassmann manifolds ($\hat{R}_{n,m} \to R_{n,m}$, $C_{n,m} \to \hat{R}_{n,m}$, $Q_{n,m} \to C_{n,m}$). Main re-

sults: If an $O'(m)$-structure $\mathfrak{F}'$ is subordinate to an $\hat{O}(m)$-structure $\hat{\mathfrak{F}}$, then

(I) $W_2(\hat{\mathfrak{F}}, t) = C_2(\mathfrak{F}', t)$,

(II) $P_0(\hat{\mathfrak{F}}, t) = C_0(\mathfrak{F}', t) \cup C_0(\mathfrak{F}', it)$,

(III) $X_0^m(\hat{\mathfrak{F}}) = (-1)^{m'} C_0^m(\mathfrak{F}')$;

here W_2 is the Stiefel-Whitney polynomial, C_2 is the Chern polynomial reduced mod 2, P_0 is the Pontryagin polynomial, C_0 is the Chern polynomial, X_0^m is the Euler-Poincaré class, $m' = m/2$ (X_0^m reduced mod 2 is the Stiefel-Whitney class W_2^m). Similar results pertain to the reductions $O(m) \to \hat{O}(m)$ and $O'(m) \to O''(m)$. Other theorems describe the effect of changes of orientation (Pontryagin), passage to a complex conjugate bundle in case (3), and duality relations for the characteristic classes of the "dual" of a given bundle (obtained by composing the defining map of the base space into, e.g., $R_{n,m}$ with the duality map $R_{n,m} \to R_{m,n}$).

In Chapter III some of the lower homotopy groups of the Grassmann manifolds are determined, using the connection with Stiefel manifolds. As an application the homotopy classes of maps of a 4-complex into $Q_{n,4}$ and $C_{n,4}$ are classified in terms of a few homology invariants. Chapter IV studies the existence of almost complex and quaternion structures in the tangent bundle of a manifold. The results of Chapters II and III give a complete classification of almost complex structures on 4- and 6-manifolds, in terms of cohomology conditions involving the characteristic classes of the tangent bundle. As a simple application of result (II) of Chapter II it is shown that the spheres S^{4n} do not admit almost complex structures.

Chapter V is concerned with the operation of product of two sphere bundles (with the new fiber being the join of the two old ones), introduced by Whitney; the main results are duality theorems which express the Stiefel-Whitney, resp. Chern, resp. quaternion, polynomials of the product as tensor products of those of the factors. By means of the diagonal map one gets the Whitney duality theorem and the duality theorem for Chern and quaternion classes for the $\cup$-product of two bundles over the same base. A duality theorem for Pontryagin classes is proved only in the case where one of the factors is simple [complete duality, with rational coefficients, or with integral coefficients but mod torsion, has been given by Chern (Topics in differential geometry, Princeton notes, 1951)]. A number of examples and theorems of more special nature are treated.

Some of the results have been announced earlier [C. R. Acad. Sci. Paris **227**, 582–584, 1076–1078 (1948); **228**, 972–973 (1949); these Rev. **10**, 203, 318, 558]. It seems regrettable that this thesis [Strasbourg, 1949] took so long to appear in print, particularly in view of the progress which has taken place in the meantime (due to the author and others). *H. Samelson* (Princeton, N. J.).

***Reeb, Georges. Sur certaines propriétés topologiques des variétés feuilletées.** Actualités Sci. Ind., no. 1183 = Publ. Inst. Math. Univ. Strasbourg **11**, pp. 91–154, 157–158. Hermann & Cie., Paris, 1952.

The object of this paper (the author's thesis, Strasbourg, 1948) is the study of the topological properties of the integral manifolds of a completely integrable Pfaffian system (=system of total differential equations) on a differentiable manifold. Some of the results have been announced earlier [C. R. Acad. Sci. Paris **220**, 236–237 (1945); **222**, 847–849 (1946); **224**, 1613–1614 (1947); **226**, 1337–1339 (1948); **228**, 47–48 (1949); these Rev. **7**, 328, 446; **8**, 595; **10**, 56, 391]. Actually the more general concept of leaved manifold (variété feuilletée) is treated: A q-leaved structure on an n-manifold V_n is given, if V_n is decomposed locally into q-cells (the elementary leaves or plaques) in just the way this happens with the integrals of a completely integrable Pfaffian system. It is then possible and useful to introduce a new topology in which the plaques are the neighborhoods; a (complete) integral manifold (=leaf) is a component of V_n in this topology. The author raises the question whether the existence in a manifold of a Pfaffian system (or a field of q-dimensional contact elements, $1 \leqq q \leqq n-1$) implies the existence of a completely integrable such system, and states his belief that the answer is negative.

Chapter A treats the more elementary properties of a leaved structure, e.g., definition of a proper leaf: the new ("inner") topology coincides with the topology inherited from V_n. Theorem: If the closure of a proper leaf is compact and contains only proper leaves, then it contains a compact leaf. Several examples are constructed, which show some unexpected properties.

Chapter B gives three stability theorems: Suppose there exists a leaf V_q, which is compact with finite fundamental group (property c). Then (I) all leaves, which come sufficiently close to V_q, have property c; (II) if the leaved structure is varied slightly, the new leaves again have property c; (III) if $q = n-1$, and V_n is compact, then all leaves have property c. Actually (II) and (III) are proved for a more general situation. In some cases the leaves are shown to be the fibers of a fiber bundle structure in V_n, e.g., for a leaved structure of class C^2 with compact, simply connected leaves.

Chapter C applies (I) and (III) to the study of the singularities of a completely integrable Pfaffian form ω, i.e., the points where the form vanishes. There is then an approximating form $\omega^* = \sum a_{ij} x_i dx_j$, whose coefficients are linear; ω^* is shown to be completely integrable, and the matrix (a_{ij}) is of rank 2 or symmetric. If the form $a_{ij}x_ix_j$ is positive definite, the singularity is called a center. It is shown that the integral manifolds near a center are topological spheres. If a form on a compact V_n ($n \geqq 3$; $n = 2$ plays an exceptional role in this chapter) has only isolated singularities, which are centers, then all integral manifolds are spheres, there are just two singularities, and V_n itself is a sphere. Analytic forms are studied in more detail; the first stability theorem is proved again for the product of a compact manifold and the real line; an existence theorem for analytic solutions of the Pfaffian is established, leading to a classification of the singular points by the index of inertia of the quadratic term.

H. Samelson (Princeton, N. J.).

Harary, Frank, and Norman, Robert Z. The dissimilarity characteristic of Husimi trees. Ann. of Math. (2) **58**, 134–141 (1953).

The dissimilarity characteristic, defined by Otter for trees [Ann. of Math. (2) **49**, 583–599 (1948); these Rev. **10**, 53] is extended to the case of Husimi trees (connected graphs in which no line lies on more than one cycle), and is shown to have always the value one. This result is the basis for the calculation of the number of Husimi trees, discussed in the last section of the paper and more fully in a previous paper by F. Harary and G. E. Uhlenbeck [Proc. Nat. Acad. Sci. U. S. A. **39**, 315–322 (1953); these Rev. **14**, 836].

L. Van Hove (Princeton, N. J.).

GEOMETRY

***Hobson, E. W., Hudson, H. P., Singh, A. N., and Kempe, A. B. Squaring the Circle and other monographs.** Chelsea Publishing Company, New York, N. Y., 1953. iv+57 pp.; iv+143 pp.; vii+110 pp.; iii+51 pp. $3.25.

This book reproduces by photo-offset the following books: E. W. Hobson, Squaring the circle [Cambridge, 1913]; H. P. Hudson, Ruler and compasses [Longmans Green, London, 1916]; A. N. Singh, The theory and construction of non-differentiable functions [Lucknow Univ. Press, 1935]; and A. B. Kempe, How to draw a straight line; a lecture on linkages [Macmillan, London, 1877].

Hjelmslev, Johannes. La géométrie sensible. III. Enseignement Math. **39** (1942–1950), 210–236 (1953).

[For parts I and II see Enseignement Math. **38**, 7–27 (1940); **38**, 294–322 (1942); these Rev. **1**, 322; **4**, 166.] An outline of the author's ideas on the teaching of solid geometry, from the simplest concepts like cubes, prisms and their volumes to rather advanced vector calculus and spherical geometry. *H. Busemann* (Los Angeles, Calif.).

Frostman, Otto. A theorem of Fáry with elementary applications. Nordisk Mat. Tidskr. **1**, 25–32, 64 (1953). (Swedish. English summary)

The theorem of Fáry [Bull. Soc. Math. France **77**, 128–138 (1949); these Rev. **11**, 393] states that the average value of the projections of an angle on the different planes through the vertex equals the angle. The applications are of the following type: In an n-sided convex polyhedral angle the sum of the edge angles is greater than $(n-2)\pi$ and less than $(n-2)\pi$ plus the sum of the face angles.

H. Busemann (Los Angeles, Calif.).

Bottema, O. A mapping of the polar triangle of a conic section on points of space. Simon Stevin **29**, 131–139 (1952). (Dutch)

Tárczy-Hornoch, A. Über den Azimutunterschied konjugierter Normalschnitte auf dem Ellipsoid. Acta Tech. Acad. Sci. Hungar. **6**, 189–199 (1953). (Russian summary)

Normal sections on an ellipsoid erected at two points and passing through the opposing points do not coincide. The azimuthal difference at either point is formulated by direct trigonometric analysis of the spheroidal geometry. Results are consistent with the corresponding presentation of Jordan and Eggert [Handbuch der Vermessungskunde, v. III/2, 8th ed., Metzler, Stuttgart, 1941, pp. 13–16].

N. A. Hall (Minneapolis, Minn.).

Muracchini, Luigi. Osservazioni sulle trasformazioni puntuali analitiche fra spazi euclidei. Boll. Un. Mat. Ital. (3) **8**, 40–43 (1953).

The following theorems are proved. a) An analytic conformal point-transformation between two euclidian planes keeping the areas is a similitude. b) An analytic point-transformation between two euclidian spaces keeping the areas is a similitude. *E. Bompiani* (Rome).

***McCrea, William H. Analytical geometry of three dimensions.** 2d ed. Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, N. Y., 1953. vii+144 pp. $1.25.

Matsumoto, Natsuo. On some projective properties. I. Rep. Univ. Electro-Commun. **1950**, no. 1, 83–101 (1950). (Japanese)

Matsumoto, Natsuo. On some projective properties. II. Rep. Univ. Electro-Commun. **1950**, no. 2, 279–295 (1950). (Japanese)

Matsumoto, Natsuo. On some projective properties. III. Rep. Univ. Electro-Commun. **1951**, no. 3, 75–91 (1951). (Japanese. English summary)

Matsumoto, Natsuo. On some projective properties. IV. Rep. Univ. Electro-Commun. **1952**, no. 4, 71–99 (1952). (Japanese. English summary)

In these papers, the author states 130 theorems on projective properties of curves and surfaces of the second order and of the second degree. The author derives these theorems by analytical methods using homogeneous coordinates. To cite a typical example: There are a conic σ and a $(2n+1)$-gon $A_1A_2\cdots A_{2n+1}$. We denote the poles of the sides A_1A_2, A_2A_3, $\cdots$, $A_{2n}A_{2n+1}$, $A_{2n+1}A_1$ with respect to σ respectively by $\bar{A}_1$, $\bar{A}_2$, $\cdots$, $\bar{A}_{2n+1}$. If $2n$ straight lines $A_1\bar{A}_{n+1}$, $A_2\bar{A}_{n+2}$, $\cdots$, $A_{2n}\bar{A}_{n-1}$ (where $A_{2n+1+k}=A_k$, $\bar{A}_{2n+1+k}=\bar{A}_k$) are concurrent at a point A and the straight lines A_1A_2, A_2A_3, $\cdots$, $A_{2n+1}A_1$ do not pass through the point A, then the straight line $A_{2n+1}\bar{A}_n$ passes through the point A. *K. Yano.*

Ayrault-Curie, G. L'orientation considérée indépendamment des axiomes de congruence et de parallélisme. Rev. Gén. Sci. Pures Appl. **60**, 68–78 (1953).

Assume the axioms of incidence of a 3-dimensional geometry (including the existence of at least four points on a line) and "between" as a primitive notion. The author investigates the notion of "positive" or "negative" correspondence of ordered simplices in the same flat. She defines the correspondence of $A, B, C, \cdots$ and $A', B, C, \cdots$ to be negative if the flat spanned by $B, C, \cdots$ contains a point between A and A', positive otherwise. The correspondence between A, B, C and $A', B', C', \cdots$ may be obtained as a composition of correspondences of the former type, and its sign is defined as the product of the signs of the factors in the composition. This definition is validated with the help of two axioms (one for the plane and one for the line), and other properties are established. It is shown how these axioms imply and are implied by the usual properties of betweenness. *A. J. Hoffman* (Washington, D. C.).

Convex Domains, Extremal Problems, Integral Geometry

Ohmann, D. Ein vollständiges Ungleichungssystem für Minkowskische Summe und Differenz. Comment. Math. Helv. **27**, 151–156 (1953).

Let A, B be convex domains in the euclidean plane with $B \subset A$. Denote the areas of A, B, $A+B$, $A-B$ by x_1, x_2, x_3, x_4, respectively. Then $x_1 \geqq x_4 \geqq 0$, $x_2 \geqq 0$, $x_3+x_4 \geqq 2(x_1+x_2)$, $2(x_1x_4)^{1/2} \leqq 3x_1+x_2-x_3$. Conversely, if 4 numbers x_1, x_2, x_3, x_4 satisfy these inequalities, then convex domains $A \supset B$ exist such that the x_i are the above areas. *H. Busemann.*

Hadwiger, H. Additive Funktionale k-dimensionaler Eikörper. I. Arch. Math. **3**, 470–478 (1952).

A functional $\varphi(A)$ defined for all convex bodies A in k-dimensional Euclidean space is called additive if for a

convex body $A+B$ formed by two convex bodies A and B whose intersection AB lies in a $(k-1)$-dimensional plane, the relation $\varphi(A)+\varphi(B)=\varphi(A+B)+\varphi(AB)$ holds, and continuous if $A_n \to A$ (in the usual sense of the theory of convex bodies) implies $\varphi(A_n)\to\varphi(A)$. If $\varphi(A)$ is, in addition, invariant under motions of A, then $\varphi(A)$ is a linear combination of the Minkowski functionals $W_\nu(A)$ $(\nu=0, 1, \cdots, k)$ defined by the formula $V(A_\rho)=\sum_0^k \binom{k}{\nu} W_\nu(A)\rho^\nu$ for the volume of the body A_ρ consisting of all points whose distance from A is $\leqq\rho$. The author had previously considered the case $k=3$ [Monatsh. Math. 54, 345–353 (1950); Abh. Math. Sem. Univ. Hamburg 17, 69–76 (1951); these Rev. 12, 734, 851]. *B. Jessen* (Copenhagen).

Debrunner, H. Translative Zerlegungsgleichheit von Würfeln. Arch. Math. 3, 479–480 (1952).

Adding to a result of Hadwiger [Collectanea Math. 3, 11–23 (1950); these Rev. 13, 768] the author proves for any two k-dimensional congruent cubes P and Q the existence of decompositions $P=\sum_1^n P_\nu$ and $Q=\sum_1^n Q_\nu$ such that P_ν and Q_ν are polyhedra congruent by translation and $n\leqq 5^{\binom{k}{2}}$. *B. Jessen* (Copenhagen).

Motzkin, T. S., Straus, E. G., and Valentine, F. A. The number of farthest points. Pacific J. Math. 3, 221–232 (1953).

A range of general theorems is developed concerning closed convex curves, each point of which has a single point on the curve farthest from it. Furthermore, sets of the Euclidean plane are studied, each point of which has at least two points on the set farthest from it. It is shown, among others, that a compact set having the above property lies in the union of a finite number of line-segments.

L. Fejes Tóth (Veszprém).

***Fejes Tóth, László. Über das Problem der dichtesten Kugellagerung.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 619–642. Akadémiai Kiadó, Budapest, 1952. (Hungarian and German. Russian summary)

Every packing of equal solid spheres in ordinary space determines a honeycomb whose cells surround the spheres as follows: each cell (or Dirichlet region) is a polyhedron whose interior consists of all points that are nearer to the center of one sphere than to the center of any other. The density D of the packing is the ratio of the volume of the sphere to the average volume of a cell. The author recalls that there are infinitely many packings of density $D_0=\pi/3\sqrt{2}=0.74048\cdots$, the cells being rhombic dodecahedra, or trapezo-rhombic dodecahedra, or some of each. We still do not know whether it is possible to pack the whole space more densely. Blichfeldt's inequality $D<0.835$ was improved by Rankin [Ann. of Math. (2) 48, 1062–1081 (1947); these Rev. 9, 226], who found $D<0.828$. But before then the author already had a "tolerably satisfactory" proof that $D<0.7545\cdots$. This upper bound, differing from D_0 by only 2%, is the value obtained by taking the cell to be a regular dodecahedron. In the present paper (which is beautifully illustrated), the author simplifies his proof and makes it still more satisfactory. He also considers a sphere surrounded by twelve equal spheres, all touching it, and asks how close to the first sphere a fourteenth equal sphere can be brought [cf. Schütte and van der Waerden, Math. Ann. 125, 325–334 (1953); these Rev. 14, 787]. His solution seems to have been superseded by Boerdijk [Philips Research Rep. 7, 303–313 (1952); these Rev. 14, 310].

H. S. M. Coxeter (Toronto, Ont.).

Few, L. The double packing of spheres. J. London Math. Soc. 28, 297–304 (1953).

A system of spheres of equal radii is said to form a packing (or a double packing), if no point of space belongs to two (or three) different spheres of the system. Let δ and δ_2 be the densities of the closest packing and double packing of spheres in R_n. The author shows that the sum of the squares of the distances between any four centres of a double packing of spheres of radius 1 is at least 16. By use of this result and Blichfeldt's density method for estimating δ, he shows that $\delta_2\leqq(n+2)(3/4)^{n/2}$. The author shows that, if the radii of the spheres of a packing are increased by the factor $(4/3)^{1/2}$, then the enlarged spheres form a double packing. He deduces that $\delta_2\geqq(4/3)^{n/2}\delta>2(1/3)^{n/2}$. *C. A. Rogers.*

Busemann, Herbert. Volume in terms of concurrent cross-sections. Pacific J. Math. 3, 1–12 (1953).

Let M_i $(i=1, 2, \cdots, n-1)$ be $n-1$ domains in the n-dimensional euclidean space E_n and let us introduce the following notations: $|M_i|$ = volume of M_i; $M_i(u)$ = cross-section of M_i with the hyperplane $H(u)$ through the fixed point z normal to the unit vector u; $dV_{p_i}^{n-1}=(n-1)$-dimensional volume element of $M_i(u)$ at the point p_i; $T(p_1, \cdots, p_{n-1}, z)$ = volume of the simplex with vertices $p_1, p_2, \cdots, p_{n-1}, z$; Ω_n = surface of the unit sphere in E_n; $d\omega_u^n$ = area element of Ω_n at the point u. Following methods of integral geometry, the author proves the integral formula

$$|M_1|\cdots|M_{n-1}| = \tfrac{1}{2}(n-1)!\int_{\Omega_n}\Big(\int_{M_1(u)}\cdots\int_{M_{n-1}(u)} T(p_1, \cdots, p_{n-1}, z) \times dV_{p_1}^{n-1}\cdots dV_{p_{n-1}}^{n-1}\Big)d\omega_u^n.$$

From this result, if M_i are convex bodies in E_n $(n\geqq 3)$ with interior points, Steiner's symmetrization leads to the following interesting inequality

$$|M_1|\cdots|M_{n-1}| \geqq \frac{1}{n}\frac{\kappa_n^{n-2}}{\kappa_{n-1}^n}\int_{\Omega_n}|M_1(u)|^{n/(n-1)}\cdots|M_{n-1}(u)|^{n/(n-1)}d\omega_u^n,$$

where $\kappa_n=\pi^{n/2}\Gamma(\frac{1}{2}n+1)$; the equality sign holds only when the M_i are homothetic ellipsoids with center z. Some generalizations, consequences, and unsolved problems are indicated. *L. A. Santaló* (Buenos Aires).

Rodríguez Vidal, R. Pfaffians and the support function in the study of convex bodies. Revista Acad. Ci. Zaragoza (2) 7, no. 1, 79–94 (1952). (Spanish)

K désignant la surface d'un corps convexe, O un point intérieur à K centre d'une sphère-unité U, considérons la normale issue de O au plan tangent à K en x. Cette normale coupe U en un point ξ de coordonnées cartésiennes (ξ_1, ξ_2, ξ_3) relativement à un repère d'origine O. Minkowski a introduit, sous le nom de fonction-support de K, la fonction $h(\xi)$ définie sur U, distance du centre O au plan correspondant tangent à K. L'auteur se préoccupe de déterminer en fonction de h et de ses dérivées partielles les principaux éléments géométriques de K, en utilisant les techniques de calcul différentiel extérieur et du repère mobile d'Elie Cartan et de

retrouver ainsi certains résultats de géométrie intégrale de Blaschke. Comme l'on sait, le cas où h est une fonction homogène de degré n des coordonnées de ξ, et particulièrement le cas $n=1$, sont d'un spécial intérêt en théorie des corps convexes.

Dans ces cas, l'auteur calcule la courbure totale et détermine aire et volume de K à l'aide de h et de ses dérivées des deux premiers ordres. Il retrouve les formules de Minkowski qui expriment $\Delta_1 h$ et $\Delta_2 h$, et, pour l'intégrale de la courbure moyenne sur K, la formule $M=2\int h dS$ où dS est l'aire de la représentation sphérique. La dernière partie du papier est consacrée à retrouver, dans cette technique, la formule de Blaschke qui exprime l'aire de K à l'aide seulement de h et $\Delta_2 h$. *A. Lichnerowicz* (Paris).

Fast, H., et Götz, A. Sur l'intégrabilité riemannienne de la fonction de Crofton. Ann. Soc. Polon. Math. **25** (1952), 309–322 (1953).

For real p and for $0 \leqq \theta < \pi$ let $N_K(p, \theta)$ denote the number of intersections of a plane curve K with the line $x \cos\theta + y \sin\theta = p$. According to Crofton the double integral in (p, θ) of $N_K(p, \theta)$ is twice the length of K. The authors give necessary and sufficient conditions for $N_K(p, \theta)$ to be Riemann integrable and hence answer in the negative a question raised by Steinhaus. *L. C. Young.*

Algebraic Geometry

Hutcherson, W. R. Invariant curves of order eight. Univ. Nac. Tucumán. Revista A. **9**, 13–14 (1952).

Sz.-Nagy, Gy. Ableitung einer algebraischen Kurve vierter Ordnung vom Index 2 aus Kegelschnitten. Publ. Math. Debrecen **2**, 166–168 (1952).

The "index" of a curve in the real projective plane is the smallest number of points in which the curve is met by a line. The author uses two pairs of properly chosen conics to construct a quartic of index 2, with one even branch and two odd branches through a double point of the quartic. *G. B. Huff* (Athens, Ga.).

Bagchi, Hari das, and Chaki, Manindra Chandra. Note on autopolar plane cubics. Rend. Sem. Mat. Univ. Padova **21**, 316–334 (1952).

It is shown that a plane cubic is its own polar reciprocal with respect to some conic if and only if the cubic is cuspidal and that for each cuspidal cubic there is a one-parameter family of conics with the desired property. The group of all collineations (and a set of quadratic transformations) leaving a cuspidal cubic invariant is exhibited. *G. B. Huff.*

Bagchi, Hari das, and Mukherji, Biswarup. Note on certain remarkable types of curves, surfaces and hypersurfaces. Rend. Sem. Mat. Univ. Padova **21**, 395–405 (1952).

Geometric characterizations are given for loci whose Cartesian equations are of the form $r^{2m}+f=0$, where r^2 is the sum of the squares of the coordinates and f is a polynomial of degree less than $2m$. It is shown that for all such loci, the algebraic sum of the eccentric angles of the $4m$ points of intersection with an arbitrary ellipse is an even multiple of π. *G. B. Huff* (Athens, Ga.).

Fabricius-Bjerre, Fr. An elementary (6, 1)-transformation. Mat. Tidsskr. B. **1952**, 1–13 (1952).

In this paper the author, starting from a non-parabolic conic C in a euclidean or non-euclidean plane, studies the transformation (6, 1), obtained by associating to a point P the point P' in which the normals to C in the points of contact of the tangents from P to C intersect. *M. Piazzolla Beloch* (Ferrara).

Brennan, J. G. Manifolds whose points image pairs or trios of points chosen from the groups of a linear series. Quart. J. Math., Oxford Ser. (2) **4**, 41–53 (1953).

The author considers a linear series g_n^r on a curve C of general genus in $[k]$, $r<k\leqq n$, and denotes by $V_r(k, n)$ those manifolds whose points P_i image the groups of k points chosen from groups of g_n^r (the point P_i being the intersection of the osculating primes of C at these points). This paper is concerned with the case of curves $V_1(2, n)$, and surfaces $V_2(3, n)$. The author calculates genus and order of a $V_1(k, n)$, the number of multiple points of $V_1(2, n)$, the intersections of $V_1(2, n_1)$ and $V_1(2, n_2)$. Hence he calculates the order of a $V_2(3, n)$ and finds the complete intersection of $V_2(3, n_1)$ and $V_2(3, n_2)$ and determines besides the multiple points and multiple curves on $V_2(3, n)$. *M. Piazzolla Beloch.*

Galafassi, Vittorio Emanuele. Di alcuni legami fra la geometria differenziale proiettiva e la geometria algebrica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 365–388 (1951).

Links between differential and algebraic geometry of plane algebraic curves. Making use of known invariants of a configuration of differential elements, keeping some of these fixed and assuming a variable element on an algebraic curve, it is possible to define on this a linear series of points, niveau-groups of a rational function on the curve, this function being one of those invariants (or a suitable power of it).

This general idea is illustrated by the following examples.

1) Differential element of the first order E_1 variable on a curve C^n and three given points (elements of order zero); the invariant is the cross-ratio of the tangent to E_1 and of the joins of its center with the fixed points.
2) Differential element E_2 varying on C^n and a fixed E_2^*; the invariant of E_2, E_2^* (Bompiani) can also be interpreted as a cross-ratio.
3) Given a curve C^n and a net (linear system ∞^2) of curves C^m, an element E_2 of C^n and the element E_2^* of the curve of the net containing E_1 have an invariant (Mehmke-C. Segre): the linear series becomes undetermined if C^n is an envelope of constant invariant of the net (Bompiani). More definite results for $m=2$.
4) Given two pencils of curves, C^n, $C^{n'}$, consider the locus Ψ of points of contact of curves of the pencils: the contact invariant of these curves at a point of Ψ defines a linear series on it.

In each case the order of the series and a pencil determining it on the curve is given. These ideas can also be used to investigate algebraic correspondences between algebraic curves. *E. Bompiani* (Rome).

Spampinato, Nicolò. Le varietà dell'S_5 complesso determinate da una curva algebrica piana. Ricerca, Napoli **3**, no. 3–4, 3–16 (1952).

With an irreducible curve c^n of equation

(1) $$f(x_1, x_2, x_3)=0,$$

in the complex projective plane, the author associates the variety $V_3^{n^2}$ defined in complex projective S_5 by equations (1) and (2): $y_1f_1+y_2f_2+y_3f_3=0$, where $f_i=\partial f/\partial x_i$. He then shows how the projective properties of c^n are exhibited by varieties related in a natural way to this $V_3^{n^2}$. For example, a suitably chosen S_3 cuts from the V_4^n of equation (2) a rational hypersurface V_2^n which is the image of the plane $y_1=y_2=y_3=0$ under a birational transformation and contains a curve V_1^η, the image of c^n under this transformation, with a single multiple point of multiplicity equal to the class m of c^n and whose order η is the sum of the order n and the class m of c^n. *G. B. Huff* (Athens, Ga.).

Togliatti, Eugenio G. Superficie algebriche ed equazioni di Laplace. Atti Accad. Ligure 9 (1952), 136–154 (1953).

A review of the known algebraic surfaces that represent an equation of Laplace; and determination of all such surfaces that are rational and that can be represented on a plane by a linear system of curves of the third degree.

Author's summary.

*Burau, Werner. **Lo spezzamento dello spazio delle relazioni quadratiche d'una grassmanniana.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 269–270. Casa Editrice Perrella, Roma, 1953.

*Godeaux, Lucien. **Involutions cycliques de genres un appartenant à une surface algébrique.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 339–349. Casa Editrice Perrella, Roma, 1953.

de Boer, J. H. The relation between the Cayley form and the Barsotti form of an algebraic chain. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 158–161 (1953).

V is a variety of dimension d in projective space S_n. The condition that a specialised $(n-d-1)$-space given by the equations $\sum_{j=0}^{n}u'_{ij}x_j=0$ $(i=0,\cdots,d)$ should meet V is given by the vanishing for $u_i=u_i'$ of the Cayley (zugeordnete) form of V

$$F(u_0, u_1, \cdots, u_d). \tag{1}$$

F can be expressed as a polynomial in the dual Grassmann coordinates of the $(n-d-1)$-space:

$$F(u_0, u_1, \cdots, u_d)=G(\cdots, p^{i_0\cdots i_d}, \cdots).$$

If we write

$$p^{i_0\cdots i_d}=\sum_{k=0}^{n-d-1}(-1)^k x_{j_k}p_{j_0\cdots j_{k-1}j_{k+1}\cdots j_{n-d-1}},$$

where $i_0, \cdots, i_d, j_0, \cdots, j_{n-d-1}$ is an even permutation of $0, \cdots, n$, and $(\cdots, p_{i_0\cdots i_{n-d-2}}, \cdots)$ are the Grassmann coordinates of a generic $(n-d-2)$-space, the equation

$$G\left(\cdots, \sum_{k=0}^{n-d-1}(-1)^k x_{j_k}p_{j_0\cdots j_{k-1}j_{k+1}\cdots j_{n-d-1}}, \cdots\right)=0 \tag{2}$$

gives the cone projecting V from the generic $(n-d-2)$-space. [For full details and references see Hodge and Pedoe, Methods of algebraic geometry, vol. II, Cambridge, 1952; these Rev. 13, 972.] If the cone given by (2) is cut with a generic $(d+1)$-space in which the coordinates are $(y_0, y_1, \cdots, y_{d+1})$, we obtain a variety ψ given by a single equation

$$\psi(y_0, \cdots, y_{d+1})=0. \tag{3}$$

The relation between corresponding points x of V and y of ψ is a birational one, and in fact $y_i=\sum_{j=0}^{n}v_{ij}x_j$ $(i=0, \cdots, d+1)$, where the v_{ij} are independent indeterminates over the ground field.

This method of derivation shows that there is an intimate connection between the Cayley form of V and the form ψ. But, unlike the Cayley form, the form ψ is not new in algebraic geometry. It has been frequently used, for example, by Severi, who ascribes it to Kronecker. It is therefore acutely embarrassing to many parties that in the paper under review ψ is called the Barsotti form, especially as Barsotti stakes no claim to it in his paper [Barsotti, Ann. of Math. (2) **52**, 427–464 (1950); these Rev. **12**, 200].

By an ingenious direct method de Boer obtains the relations between the Cayley form and ψ, and shows that the coefficients of ψ arise from the coefficients of the Cayley form by a projective transformation defined over the prime field. The form ψ is therefore not essentially different from the Cayley form, and may be used to attach coordinates to chains (multiplicative varieties, cycles) in S_n.

D. Pedoe (Khartoum).

Roth, Leonard. On threefolds of linear genus unity. Ann. Mat. Pura Appl. (4) **34**, 247–276 (1953).

This paper is devoted to a survey of the algebraic threefolds of absolute linear genus unity. The author divides such threefolds into four main classes: (I) threefolds whose canonical and pluricanonical systems are all virtual; (II) threefolds possessing isolated (pure) canonical and pluricanonical surfaces (possibly of order zero); (III) threefolds with effective canonical or pluricanonical systems compounded of a pencil of surfaces of zero virtual grade; (IV) threefolds whose canonical or pluricanonical system is compounded of a congruence of elliptic curves. It is assumed that, if the general canonical (or pluricanonical) surface is irreducible, it is simple and that if it is reducible it is free from fixed parts. The bulk of the paper is concerned with the construction of threefolds of these types and an examination of their properties. *J. A. Todd.*

Roth, L. Hyperelliptic threefolds. Proc. Cambridge Philos. Soc. **49**, 397–409 (1953).

Let V be a nonsingular algebraic threefold admitting a continuous permutable group $\mathfrak{G}$ of ∞^2 automorphisms. The trajectories of this group form a pencil, of genus $\rho\geqq 0$, of surfaces C. It is assumed that the generic C is irreducible, and that $\mathfrak{G}$ acts transitively and without exceptions on C, and that C represents in an unexceptional (1-1) manner the transformations of $\mathfrak{G}$. Then C is a nonsingular hyperelliptic surface free from exceptional curves. Then V contains a hyperelliptic congruence Γ of birationally equivalent curves $\mathfrak{C}$. The intersection number $d=[\mathfrak{C}C]$ is called the determinant of V. Each curve $\mathfrak{C}$ possesses a permutable group of birational self-transformations (which imposes a restriction on its moduli if its genus π exceeds 2, and which implies that, for $\pi\geqq 1$, d can take only a finite number of values). If $d=1$, V is birationally equivalent to the product $\mathfrak{C}\times C$, and $\pi=\rho$. If $d>1$, V can be mapped on the d-fold threefold $V^*=\mathfrak{C}^*\times C^*$, where $\mathfrak{C}^*$ is a curve of genus ρ birationally equivalent to the pencil $\{C\}$, and C^* is a surface birationally equivalent to the congruence Γ. In general, the only reducible elements of $\{C\}$ are a certain number of surfaces C each of which is an irreducible surface C_s counted s times; C_s is a hyperelliptic surface in $(d/s, 1)$ correspondence with Γ; the image of C_s on V^* is an $(s-1)$-fold component of the

branch surface of the $(d, 1)$ correspondence between V and V^*. The characters $s_1, \cdots, s_N$ corresponding to the various multiple elements in $\{C\}$ satisfy

$$2(\pi-1)-2d(\rho-1)=d\sum(s_i-1)/s_i.$$

The author observes incidentally that if two of the s_i are equal, V has divisor $\sigma>1$. Under the assumption that V^* is free from exceptional surfaces, the author shows that the canonical system on V is equivalent to $\sum(s-1)C_s+(2\rho-2)C$ and computes the invariants of V; in particular, $\Omega_1=P_0=1$. Some particular cases are examined; and the author proves that $P_{12}=0$ is a necessary and sufficient condition for $\pi=0$. Analytic representations are obtained for cyclic threefolds.

J. A. Todd (Cambridge, England).

Villa, Mario. Sulle trasformazioni di contatto algebriche. Boll. Un. Mat. Ital. (3) **8**, 6–9 (1953).

Let $F(x, y, \bar{x}, \bar{y})=0$ determine a contact-transformation between two planes $\pi_1(x, y)$, $\pi_2(\bar{x}, \bar{y})$.

If F is a polynomial, to the points of π_1 (or $\bar{\pi}$) correspond ∞^2 algebraic curves in $\bar{\pi}$ (or π): let Σ_1, Σ_2 be the minimum linear systems containing the two systems of ∞^2 curves in π_1 and π_2. The systems Σ_1, Σ_2 have the same dimensionality δ: when a point describes a curve C of one of the two systems, the corresponding curve, determined by $F=0$, describes a system ∞^1 which is contained in a minimum linear system determined by C. Let us call the correspondence between C and the linear system a reciprocity R. The following theorem is proved. Given in π, a differential element of the first order E_1 (point P and tangent through it), consider: 1) the system $\infty^{\delta-1}(\delta>1)$ of Σ_1 through P and the curve C corresponding to it (by R); 2) the system $\infty^{\delta-2}$ of Σ_1 containing E_1 and the corresponding pencil of curves, F (by R); 3) the system $\infty^{\delta-3}$ of Σ_1 having at P a double point (at least) and the corresponding net Δ (by R). The elements corresponding to the given E_1 (by the contact transformation) belong to C and have their centers at the base-points of F which are not base-points of Δ.

E. Bompiani.

Differential Geometry

Marchaud, André. Sur les propriétés différentielles du premier ordre des surfaces d'ordre borné et plus particulièrement de celles du troisième ordre. Ann. Sci. Ecole Norm. Sup. (3) **69**, 303–370 (1952).

A simple half-cone $\mathfrak{D}$ with vertex O consists of the rays from O to the points of a simple closed spherical curve about O. The conical order of $\mathfrak{D}$ is the maximum number of rays along which $\mathfrak{D}$ is intersected by any plane that does not have a whole sector in common with $\mathfrak{D}$. It is even if it exists. If $\mathfrak{D}$ has the conical order two, then $\mathfrak{D}$ is either a plane through O or a pair of non-opposite half-planes with a common axis through O ["dihedron"] or $\mathfrak{D}$ is convex.

A surface (S) is defined by means of a function $z=f(x, y)$ which is continuous in a closed convex domain (C) in the (x, y)-plane. Let M, M', M'' denote points of (S). A [half-] tangent at M is the limit of straight lines $M'M''$ [of rays MM'] as M', M'' converge to M. $\mathfrak{T}(M)$ = paratingent at M [= set of all tangents at M]; $\mathfrak{D}(M)$ = contingent at M [= set of all half-tangents at M]. Thus $\mathfrak{T}(M)$ and $\mathfrak{D}(M)$ are closed and $\mathfrak{D}(M)\subset\mathfrak{T}(M)$. An interior point M is ordinary if $\mathfrak{T}(M)$ is not the whole space.

Let (S) be of bounded order, i.e., every straight line which has no segment in common with (S) intersects (S) at a bounded number of points only. Their least upper bound is the order of (S). A convex subdomain of (C) determines a subsurface of (S) whose order is not greater than that of (S). The order of a point M is the order of a sufficiently small subsurface containing M in its interior. Results: (1) The set of the ordinary points of (S) is open and everywhere dense. (2) Let M be an ordinary point; then $\mathfrak{D}(M)$ is a simple half-cone. If the order n of M is even, then $\mathfrak{D}(M)$ has a conical order $\leqq 2n-2$. If n is odd, then that order is $\leqq 2n$ and any plane through M which meets $\mathfrak{D}(M)$ along $2n$ rays contains exactly n pairs of opposite half-tangents.

Let $(S)=(S_3)$ be of order three and let M be ordinary. Thus the conical order of $\mathfrak{D}(M)$ is equal to 2, 4, or 6. In the last case (S_3) is part of a cone of order three. Let $\mathfrak{D}(M)$ have the conical order four. Then it contains not less than two pairs of opposite half-tangents. As far as they are projected into (C), they lie on (S_3). If there are more than two of them, they lie in a plane which meets $\mathfrak{D}(M)$ in two entire opposite sectors. Examples are given for all the possible types of such points. Various results are also obtained under the assumptions that M is an interior point of (S_3) and that either a vertical intersection of (S_3) has opposite half-tangents at M or that $\mathfrak{D}(M)$ is a plane.

Suppose (S_3) has nowhere a local plane of support. Then (S_3) contains no piece of a simple half-cone and not more than four interior non-ordinary points. They are linearly independent. If the segment connecting two of them does not lie on (S_3), then its complement does. As long as M is ordinary and $\mathfrak{D}(M)$ is not a dihedron, $\mathfrak{T}(M)$ is a plane varying continuously with M. Let $\mathfrak{D}(M')$ be a dihedron. Then M' is ordinary and the axis Δ of $\mathfrak{D}(M')$ lies on (S_3). It meets no other straight line contained in (S_3). If M is an interior point of (S_3), $M\subset\Delta$, $M\neq M'$, then $\mathfrak{T}(M)$ is a fixed plane through one of the two half-planes of $\mathfrak{D}(M')$. This half-plane meets (S_3) only along Δ. The points M' have no accumulation points in the interior of (S_3). If (C) is the whole plane, then (S_3) can have only one point M'. If there is one, then Δ is the only straight line on (S_3). The author gives an example of an (S_3) with a point M' and conjectures that the number of these points is always bounded.

P. Scherk (Saskatoon, Sask.).

Marchaud, André. Sur une classe de points singuliers des surfaces du troisième ordre de la géométrie finie. J. Math. Pures Appl. (9) **31**, 319–340 (1952).

A point set in the projective plane has the order n if it has not more than n points in common with any straight line, the maximum number n being reached. The straight line is a "curve of order one". An "irreducible curve of order two" is a Jordan curve C_2 of order two. An "irreducible curve of order three" C_3 consists of a closed parameter curve plus possibly a C_2 or an isolated point such that the entire C_3 has the order three. The parameter curve also has the order three. It has not more than one double point. A "reducible curve of order two" consists of two distinct straight lines or of one isolated point. A "reducible curve of order three" is the union of a straight line with a reducible or irreducible curve of order two. An "irreducible surface of order n" is a closed point set in projective space which intersects any plane in a curve of order $\leqq n$ and some plane in a C_n [$n=2, 3$]. An irreducible surface Ω of order two is known to be either an "ovoid", i.e., the projective image of a strictly convex closed surface, or a convex cone or a ruled quadric [Marchaud, same J. (9) **15**, 293–300 (1936)].

Let S, Σ denote irreducible surfaces of the third order. A "tangent" at a point M of S is any limit of straight lines through two distinct points of S converging to M. The point M is "regular" if some straight line through M meets S at exactly three points. M is "singular" if every point of some neighborhood of M on S lies on a tangent of M. Every non-regular point of S is singular [see the paper reviewed above]. There are surfaces S with regular singular points. Let S be such a surface. Then S permits one and only one decomposition $S=\Sigma\cup\Omega$. The intersection $\Sigma\cap\Omega$ is either empty, then every point of S is regular, or $\Sigma\cap\Omega$ is a point A, then A is the one and only non-regular point of S. Unless Σ and Ω are cones with the common vertex A, Ω is an ovoid. All the singular points of S lie on Σ, and Σ contains with any two such points the straight line connecting them. The set of the singular points of S consists either of not more than three points (then they are non-collinear) or it is a straight line (then Σ is a cone whose vertex lies on this line).

P. Scherk (Saskatoon, Sask.).

Zin, Giovanni. Contributo alla geometria infinitesimale diretta delle curve piane. Ann. Mat. Pura Appl. (4) **34**, 41–53 (1953).

Si introduce il concetto di ampiezza di un arco AB di una curva semplice C quale oscillazione di una determinazione continua $\alpha(P_1, P_2)$ dell'anomalia di una secante orientata P_1P_2, essendo P_1 e P_2 punti di AB con $P_1 \neq P_2$. Si definisce poi l'angolosità di C in un suo punto P quale limite dell'ampiezza di un arco P_1P_2, al quale P risulta interno, per P_1 e $P_2 \to P$. Si definiscono punti lipschitziani di C quelli in cui l'angolosità di C è $<\pi$. Si esamina la relazione fra angolosità e paratingente. Infine si stabilisce l'esistenza di curve semplici, rettificabili e prive di punti lipschitziani. (Author's summary.)

C. C. Torrance.

Mineo, Massimo. Sopra un calcolo più approssimato della deviazione della geodetica da una sezione normale su una superficie qualunque. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 255–259 (1953).

Hartman, Philip, and Wintner, Aurel. On the third fundamental form of a surface. Amer. J. Math. **75**, 298–334 (1953).

Let $g_{ik}(u^1, u^2)du^idu^k$ be positive definite in a certain simply connected (u^1, u^2)-domain, where the g_{ik} are of class C'. Then the Christoffel symbols and the geodesic curvature of a curve of class C'' are defined, but in general not the Gauss curvature. Using the Gauss-Bonnet Theorem as definition, the integral curvature (i.e., the integral over the Gauss curvature) over a domain bounded by a piecewise smooth curve can be defined. The surface has constant curvature K_0 if the integral curvature equals K_0 times area. In that case a parameter transformation of class C'' exists such that the line element takes the form

$$(du^2+dv^2)[1+(u^2+v^2)K_0/4]^{-1}.$$

Consider a surface S without parabolic points in E^3 of class C'' written vectorially as $\mathfrak{x}(u^1, u^2)$ such that $\mathfrak{x}_1\times\mathfrak{x}_2\neq 0$ where $\mathfrak{x}_i=\partial\mathfrak{x}/\partial u^i$, defined in a simply connected (u^1, u^2)-domain D. If $\Lambda=(\mathfrak{x}_1\times\mathfrak{x}_2)/|\mathfrak{x}_1\times\mathfrak{x}_2|$, we put $b_{ik}=-N_i\mathfrak{x}_k$, $c_{ik}=N_iN_k$, so that $\Phi_2=b_{ik}du^idu^k$ and $\Phi_3=c_{ik}du^idu^k$ are the second and third fundamental forms. The main result of the paper is this: If $N(u, v)$ is a C''-representation of the unit sphere (locally), then Φ_3 and Φ_3 have the following properties: $|b_{ik}|\neq 0$, Φ_3 is a metric of constant curvature; and if Γ^i_{jk} are the Christoffel symbols with respect to Φ_3, then for any piecewise smooth Jordan curve J in D and the interior E of J

$$\int_J b_{ik}du^k=\iint_E(\Gamma^j_{i1}b_{j2}-\Gamma^j_{i2}b_{j1})du^1du^2, \quad i=1, 2,$$

which is, of course, an integrated form of the Codazzi equations. Conversely, if Φ_2 and Φ_3 have all these properties, then a unique (up to motions) surface $\mathfrak{x}(u, v)$ in E^3 with $N(u, v)=(\mathfrak{x}_1\times\mathfrak{x}_2)/|\mathfrak{x}_1\times\mathfrak{x}_2|$ exists such that $N(u, v)$ provides a C''-representation of the unit sphere and $b_{ik}=-\mathfrak{x}_iN_k$, $c_{ik}=N_iN_k$.

It is also shown that a surface with a given Φ_3 and a given mean curvature H, or given Gauss curvature $K\neq 0$, or a given $2H/K$ exists. There are various other theorems covering differentiability hypotheses in classical theorems.

H. Busemann (Los Angeles, Calif.).

Hopf, Eberhard. On an inequality for minimal surfaces $z=z(x, y)$. J. Rational Mech. Anal. **2**, 519–522 (1953).

Improving a previous inequality of E. Heinz, the author derives the following result. Let $z(x, y)$ be a solution of class C'' of the equation $(1+q^2)r-2pqs+(1+p^2)t=0$ (of minimal surfaces) in a circular disc $(x-x_0)^2+(y-y_0)^2<R^2$. Then there exists a constant c (independent of R and of $z(x, y)$) such that $|K|\leqq c\zeta^2(1+\zeta^2)/R^2$, where K is the Gauss curvature of the surface $z=z(x, y)$ and $\zeta^2=1/(1+p^2+q^2)$ at the point (x_0, y_0). The proof is based on the close relationship between mimimal surfaces and analytic functions of a complex variable.

T. Radó (Columbus, Ohio).

Kruppa, E. Über die Affinnormalebenen der durch eine Kurve gelegten Zylinderflächen. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **89**, 207–210 (1952).

If a parabolic cylinder hyperosculates a given cylinder Z along a generator, the diameter plane of this parabolic cylinder is called the affine normal plane of Z for this generator. Let a curve C be projected into a curve C_1 in the osculating plane at a point P by the various cylinders through C. The paper discusses the relation between the affine normal to C_1 at P and the affine normal plane to Z along the generator through P.

S. B. Jackson.

Bartsch, Helmut. Über eine Klasse von Hyperflächengeweben. Ann. Mat. Pura Appl. (4) **34**, 349–364 (1953).

The author studies $(n+1)$-webs of hyperplanes, to be called diagonal webs, in an n-dimensional space with the property that the diagonals of every three-dimensional intersection web are straight lines. [For terminology cf. W. Blaschke and G. Bol, Geometrie der Gewebe, Springer, Berlin, 1938.] By analytical formulation previously established by the author [Ann. Mat. Pura Appl. (4) **32**, 249–269 (1951); these Rev. **13**, 775] it is proved that a diagonal web consists of $n+1$ pencils of hyperplanes, whose axes (that is, $(n-2)$-dimensional linear subspaces through which pass the hyperplanes of these pencils) lie on a hypersurface of order $n-1$. The equations of the axes and the equation of the hypersurface are determined by the values of the invariants of the web at a point. Diagonal webs whose invariants coincide at a point are projectively equivalent. This has the consequence that two diagonal webs are topologically equivalent (that is, equivalent under local general point transformations), if and only if they are projectively equivalent.

The question as to the conditions that $n+1$ pencils of hyperplanes which form an $(n+1)$-web, form a diagonal

web is answered by the result: The web is diagonal if and only if through every point of the axis of every pencil there is at least one line which intersects all other axes. The paper concludes with further results involving properties of intersection webs. *S. Chern* (Chicago, Ill.).

Čahtauri, A. I. The intrinsic geometry of plane nets. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **15**, 101–148 (1947). (Russian. Georgian summary)

The paper consists of four chapters the first of which is devoted to the intrinsic construction of a projective geometry in a plane in which there is given a correspondence between points and lines. If the points are given by $x^\alpha(u^1, u^2)$ and the corresponding lines by $\xi_\alpha(u^1, u^2)$, the point not being on the line, then the three points x^α, y_1^α, y_2^α where $y_i^\alpha = \partial_i x^\alpha - l_i x^\alpha$ with l_i so determined that $\xi_\alpha y_i^\alpha = 0$, form a moving trihedral from which an affine connection is constructed in the usual manner. It then follows that the curvature tensor of this correspondence is given by $R^m{}_{pki} = \delta_i^m P_{[pk]} - \delta^m{}_{[p} P_{k]i}$, thus showing that the connection is projectively euclidean. This construction is then dualized and the various possible relations between the two are examined.

The remaining three chapters are devoted to special correspondences—or configurations—determined by planar nets, in particular, to Laplacian, mutually Laplacian and auto-Laplacian configurations. The cases where the geometry reduces to that of Weyl or Riemann are examined. There are a few misprints and one or two unjustified assumptions. *M. Knebelman* (Pullman, Wash.).

Berezina, L. Ya. Investigation of congruences whose rays form a constant angle with the normals of the middle surfaces by means of the trihedron of S. D. Rossinskiĭ. Latvijas PSR Zinātņu Akad. Vēstis **1951**, no. 8 (49), 1317–1325 (1951). (Russian. Latvian summary)

This is the paper referred to at the end of the review of Berezina, Uspehi Matem. Nauk (N.S.) **7**, no. 3(49), 121–122 (1952); these Rev. **14**, 404. A trihedron is called after Rossinskiĭ if its point of application is at the center of the ray of the congruence, the axis of e_1 lying in the tangent plane to the middle surface and the axis of e_3 along the ray. Then

$$\omega_{13} = A\omega_1 + B\omega_2,\ \omega_{23} = B'\omega_1 - A\omega_2,\ \omega_3 = D\omega_2,\ \omega_{12} = E\omega_1 + F\omega_2.$$

In this paper the six coefficients are given a geometrical meaning, e.g.,

$$A = -\frac{1}{\rho}\frac{\sin 2\alpha}{\sin 2\psi},\quad B = -\frac{1}{\rho}\frac{\cos 2\alpha + \cos 2\psi}{\sin 2\psi},$$

$$B' = -\frac{1}{\rho}\frac{\cos 2\alpha - \cos 2\psi}{\sin 2\psi},$$

where 2ρ is the distance between the focal points, 2ψ the angle between the focal planes and α is the angle of rotation about e_3 by which we pass from a canonical trihedron ($\omega_{13} = b\omega_2$, $\omega_{23} = b'\omega_1$) to T. As to D, the surfaces mentioned in the title are determined by $dD = 0$. The equations of structure are derived for this case and attention is paid to the special case where the congruence is also normal ($B = B'$). Another special case of such a congruence has the projection of the ray on the tangent plane to the middle surface in the direction of a line of curvature. In the case that the congruence is normal the geodesic curvature of the line of curvature along which the projection lies is zero.

D. J. Struik (Cambridge, Mass.).

Coutoumanos, A. Rectilinear congruences whose developables intersect a surface in its Tchebychef curves. Bull. Soc. Math. Grèce **27**, 1–24 (1953). (Greek. English summary)

The spherical image of a developable surface is the curve intercepted on the unit sphere by its radii parallel to the generators of the surface. The following problem is shown to have solutions: given the unit sphere in a definite parametric representation, there exists a rectilinear congruence such that its developable surfaces have the parameter curves on the sphere as spherical images and intersect a suitable other surface in a Tchebychef net. The applications of this fact given in the paper are trivial, for instance, that a surface with an orthogonal Tchebychef net is developable.

H. Busemann (Los Angeles, Calif.).

Petkantschin, B. Über die isotropen Regelscharen im elliptischen Raum. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie I. **47**, 93–105 (1951). (Bulgarian. German summary)

Consider a family of isotropic lines in complex elliptic space (three-dimensional), each given by two of its points $x(u)$ and $p(u)$, where $p(u)$ lies on the absolute locus and u is a complex parameter. It is shown that on each line a point invariantly connected with the family exists and that an invariant parameter can be introduced in terms of which this point is expressed in a simple way. *H. Busemann.*

Carton, Maurice. Sur les suites de Laplace périodiques. Bull. Soc. Roy. Sci. Liège **22**, 101–111 (1953).

Les droites de l'espace projectif ordinaire S_3 sont représentées par les points d'une hyperquadrique Q d'un espace à cinq dimensions. Les tangentes asymptotiques au point générateur d'une surface (x) non réglée ont pour images sur Q deux points U, V consécutifs dans une suite de Laplace L: $\cdots U_2U_1UVV_1V_2\cdots$. Godeaux a étudié cette suite dans de nombreux mémoires et particulièrement dans son ouvrage: La théorie des surfaces et l'espace réglé, Hermann, Paris, 1934. En s'inspirant des méthodes et des résultats de L. Godeaux, l'auteur étudie le cas où la suite L supposée illimitée, est périodique, de période huit; il montre essentiellement qu'à la surface (x), on peut alors associer une autre surface qui, aux points correspondants, a les mêmes quadrilatères de Demoulin que (x). Il termine par l'étude des surfaces de Klein-Lie qui correspondent à des suites L périodiques.

Signalons que, dans une communication récente [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 363–368 (1953)], L. Godeaux a determiné le couple le plus général de surfaces ayant mêmes quadrilatères Demoulin.

M. Decuyper (Lille).

Legrain-Pissard. Sur la suite de Laplace de l'espace réglé associée à une surface. Bull. Soc. Roy. Sci. Liège **22**, 112–121 (1953).

Les notations sont les mêmes que dans le mémoire précédent, de M. Carton. Etant donné la surface (x), l'auteur lui associe quelques congruences (r), (p), (t), (e): l'image R d'une droite r de (r) est l'un des points de rencontre de U_1V_1 avec la quadrique Q; la droite $p(t)$ a pour image l'intersection $P(T)$ autre que $U(V)$ de $UU_2(VV_2)$ avec Q; l'image E d'une droite e est l'un ou l'autre des points de rencontre

de U_2U_3 avec Q. Si (x) est une surface de Klein-Lie, les congruences (r), (p), (t), (e) sont W. (Aux pages 120 et 121, on a plusieurs fois confondu le caractère v avec le caractère grec ν.) *M. Decuyper* (Lille).

Pylarinos, O. Sur une classe particulière de géodésiques des surfaces réglées W. Bull. Sci. Math. (2) 77, 63–72 (1953).

On a Weingarten surface there exists a one-parameter family of curves along which the mean and total curvatures of the surface are constant. If one of these curves is also a geodesic, then its curvature κ and torsion σ satisfy a relationship of the form $\kappa^2+\sigma^2=\lambda+\mu\kappa$, where λ and μ are constants. Such a geodesic is called a W-geodesic. The author computes the geodesic curvature directly for the two non-developable ruled W-surfaces and shows that these surfaces have precisely one W-geodesic each. Then, working with a developable surface as a rectifying developable of one of its geodesics, the author shows that: 1) the cylinder of revolution is the only developable surface which has two W-geodesics with the same constant mean curvature along both; 2) the cylinder of revolution is the only developable which has more than two W-geodesics; and 3) the curvature and torsion of any geodesic on a developable surface which admits two W-geodesics satisfy a relationship of the form $f(\kappa, \sigma/\kappa)=0$, and this relationship is algebraic if the ratio of the constant mean curvatures along the two W-geodesics is a rational number. *A. Schwartz* (New York, N. Y.).

Cherep, Rebeca. Affine invariants of pairs of space curves. Math. Notae 11, 110–123 (1951). (Spanish)

By the neighborhoods up to and including the third order of each of five different pairs of curves at two ordinary points in an ordinary space, affine invariants, which are not projective invariants, are derived, together with their metric characterizations. A typical one of the above five cases is concerned with two curves C, C' and two ordinary points A, B on C, C' respectively under the conditions that A is on the tangent t_B of C' at B and the osculating planes of C, C' at A, B do not pass through t_B and the tangent of C at A, respectively. *C. C. Hsiung* (Bethlehem, Pa.).

Dalla Volta, Vittorio. Sulle superficie di S_n possedenti un doppio sistema coniugato. Boll. Un. Mat. Ital. (3) 8, 29–36 (1953).

A synthetic proof of a theorem of the reviewer [Trans. Amer. Math. Soc. 70, 312–322 (1951); these Rev. 13, 276] is given. The doubt raised by the author in the cited review concerning the validity of this theorem is removed by observing that a proper hyperquadric in the theorem means a hyperquadric not passing through the point x.

C. C. Hsiung (Bethlehem, Pa.).

***Calapso, Renato. Questioni di geometria conforme.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 287–318. Casa Editrice Perrella, Roma, 1953.

This paper is a summary (almost without proofs) of results in conformal geometry of surfaces published by P. Calapso in the period of 1903–1932. First, three fundamental conformal invariants ω, Ω, W are introduced. They constitute a complete set of conformal invariants in the theory of surfaces (and satisfy two differential equations). Thus, for instance, the invariants of the Laplace equations (of a surface referred to parametric lines of curvatures) are expressible in terms of ω, Ω, W. On the other hand these conformal invariants play an important role in the investigation of isothermal surfaces (with the line element $e^{2\varphi}(du^2+dv^2)$). The fundamental Gauss-Mainardi-Codazzi equations lead in this case to a differential equation (of the fourth order for Ω) which is the integrability condition of two differential equations for the conformal invariant

$$-J=\varphi_{uu}-\varphi_{vv}+\varphi_u^2-\varphi_v^2-\omega\Omega.$$

This invariant is a starting point for the theory of "the transformation C_m" which carries an isothermal surface having invariants J, Ω into another isothermal surface having the invariants $J+2m$, Ω ($m\neq 0$ being an arbitrary constant). Among other applications we mention the connection between isothermal and Guichard surfaces, the relationship between projective and conformal invariants (Lie's "Kugel-Gerade" Transformation) and the conformal invariants of normal nets of curves. *V. Hlavatý* (Bloomington, Ind.).

Haimovici, Adolf. Sur la géométrie d'un groupe de transformations tangentielles. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 119–124 (1949). (Romanian and French)

The group in question is $u_i'=a_i^j u_j+a_i^0$ where a_i^j is an orthogonal matrix and the u_i are tangential plane coordinates in 3 dimensions. Certain invariants and geometric properties of one- and two-parameter families of planes are discussed with respect to this group. *J. L. Vanderslice.*

Haimovici, Adolf. On some asymptotic transformations. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 847–854 (1949). (Romanian. Russian and French summaries)

This is a study by the method of E. Cartan of transformations of surfaces which preserve asymptotic lines. Typical results are the following. A surface in asymptotic correspondence with a given surface preserving total curvature depends on four arbitrary functions of one variable. There is but one pair of surfaces in asymptotic correspondence which pass through two given curves tangent to the developables of these curves and having equal principal curvatures at corresponding points. *J. L. Vanderslice.*

Tharrats Vidal, Jesús M.ª. Dynamic study of pencils of transformations. An. Real Soc. Españ. Fis. Quim. Ser. A. Fis. 48, 17–26 (1952). (Spanish. English summary)

This is an expository article, dealing chiefly with known properties of one-parameter families of transformations (mostly projective or affine transformations) of a space into itself. Some remarks are made concerning physical applications of the theory, but the reviewer has found these extremely obscure. *L. A. MacColl* (New York, N. Y.).

Mihăilescu, Tiberiu. Correspondences in a point of a non-holonomic variety. Acad. Repub. Pop. Române. Bul. Şti. Ser. A. Mat. Fiz. Chim. 1, 947–958 (1949). (Romanian. Russian and French summaries)

The author studies certain correspondences at a point of projective 3-space with respect to a non-holonomic variety V_3^2 between the plane bundle at the point and the points of the plane tangent there to V_3^2. When the variety is holonomic they reduce to the correspondance of Segre.

J. L. Vanderslice (Silver Spring, Md.).

Martini, Giulia. Sulle trasformazioni puntuali fra due spazi nel caso conforme. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 225–232 (1949).

This is a study of point transformations between two spaces in the neighborhoods of two corresponding points

under assumption that the characteristic directions of the transformation at the point form a quadric cone, thus rendering the transformations locally conformal. By considering neighborhoods of the third order, geometric elements arise which permit fixing an intrinsic projective reference system in the two spaces. *J. L. Vanderslice.*

Ślebodziński, W. Sur les déformations de l'espace basé sur le groupe $x=hx+a$, $y=kx+h^m y+b$. Ann. Soc. Polon. Math. **25** (1952), 231–237 (1953).

On désigne par G_4 le groupe de transformations linéaires à quatre paramètres $x=hx+a$, $y=kx+h^m y+b$, et par E_2 l'espace affine à deux dimensions dont la géométrie est déterminée par G_4. On appelle déformation de l'espace E_2 une transformation ponctuelle T telle que deux morceaux infiniment petits, qui se correspondent par T, puissent se déduire l'un de l'autre, aux infiniment petits d'ordre deux près, par une transformation du groupe G_4. Les déformations de l'espace E_2 constituent un groupe infini de transformations à deux variables dépendantes de deux fonctions arbitraires d'une argument. On l'appelle groupe de déformations de l'espace E_2 et on le désigne par Γ. Les invariants, par rapport à Γ, des objets géométriques de l'espace E_2 sont appelés leurs invariants de déformation. L'auteur étudie les invariants par rapport à Γ de la famille de courbes définies par l'équation différentielle $dy/dx=f(x, y)$. *K. Yano.*

Takano, Kazuo. Homogeneous contact transformations and a metric space. Rep. Univ. Electro-Commun. **1950**, no. 2, 269–278 (1950). (Japanese)

To study invariant theory of homogeneous contact transformations, Eisenhart and Knebelman first introduced the so-called first contact frame [Ann. of Math. (2) **37**, 747–765 (1936)]. Y. Mutô and the reviewer [Tôhoku Math. J. **45**, 295–307 (1939)] and T. C. Doyle [Ann. of Math. (2) **42**, 698–722 (1941); these Rev. **3**, 19] have also studied such a theory introducing the so-called second contact frame. Eisenhart [Ann. of Math. (2) **49**, 227–254 (1948); these Rev. **9**, 380] has recently shown that the results of E. Cartan [Les espaces de Finsler, Hermann, Paris, 1934] on Finsler spaces can be derived by applying homogeneous contact transformations to the well known geometric objects and concepts of Riemannian spaces. However, he did not use the results of Y. Mutô, the reviewer and T. C. Doyle. The paper under review has the same purpose as that of Eisenhart, but the author tries to utilise the results of the above mentioned authors. *K. Yano* (Tokyo).

Davies, Evan T. On the invariant theory of contact transformations. Math. Z. **57**, 415–427 (1953).

The study of invariant theory of contact transformations by the use of tensor calculus was first carried out by T. Hosokawa [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **2**, 169–176 (1934)] and the reviewer [Proc. Phys.-Math. Soc. Japan (3) **17**, 39–47 (1935)]. But the first systematic study of the theory was given by L. P. Eisenhart and M. S. Knebelman [Ann. of Math. (2) **37**, 747–765 (1936)] who introduced the so-called first contact frame and, by virtue of this, could decompose a configuration A of $2n$ components in the phase space of $2n$ coordinates x^i and p_i into two vectors λ and μ of n components each. However, their decomposition was not complete and it was completed by Y. Mutô and the reviewer [Tôhoku Math. J. **45**, 295–307 (1939)] and T. C. Doyle [Ann. of Math. (2) **42**, 698–722 (1941); these Rev. **3**, 19] by introducing the so-called second contact frame. Doyle introduces the notion of n-dimensional non-holonomic subspaces in the $2n$-dimensional phase space.

In the paper under review, the author follows the method of Doyle. In §1, he gives a short account of the theory of non-holonomic subspaces and in §2, he considers simultaneous transformations of coordinates of the phase space and of base vectors defining the non-holonomic subspaces. In §3, the author applies these results to the spaces admitting contact transformations and shows that the so-called connecting quantities lead naturally to the two sets of contact frames and that the restrictions imposed by Doyle in determining the connection coefficients are not necessary. In the last section, the author shows how one can construct a geometry which is a generalization of the geometries based on a covariant vector, by considering the non-holonomic subspaces of a space whose tensors are functions of position only. *K. Yano* (Tokyo).

Bompiani, E. Topologie des éléments différentiels et quelques applications. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 9–36. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

For some years the author has been exploiting the notion of differential patches (calottes) of topological manifolds. In the present publication he first gives a clear brief exposition of the fundamentals of this discipline, both philosophical and technical, and then shows its wide range of applicability with a series of investigations in the fields of algebraic geometry, projective differential geometry, and partial differential equations. *J. L. Vanderslice.*

Blanuša, Danilo. Les espaces elliptiques plongés isométriquement dans des espaces euclidiens. I. Le plan elliptique. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 3–23 (1953). (Serbo-Croatian summary)

The elliptic plane—arising from the sphere by identification of diametrically opposite points—can in only one way be imbedded in R^3 in such a manner that the geodesics are circles. The interesting, almost purely synthetic, proof of this theorem is followed by a number of properties of intersections of this imbedded space with flat subspaces R^1, R^2, R^3, R^4 of R^5. *A. Nijenhuis* (Princeton, N. J.).

Yanenko, N. N. Some questions of the theory of imbedding of Riemannian metrics in Euclidean spaces. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 21–100 (1953). (Russian)

A rather long, mostly expository, paper dealing with the problem of imbedding a Riemann space V_m in Euclidean space E_N. About half of the paper is devoted to the algebra of skew-symmetric forms and to exterior differential forms which were used so effectively by Cartan in all sorts of problems of differential geometry. The author carefully proves even the classical theorems such as the necessity and sufficiency of the equations of Gauss and Peterson-Codazzi for hypersurfaces. After defining the rank and type of a Riemann space the author obtains the necessary and sufficient conditions that a V_m should be embeddable in E_{m+r}. He also proves that for a V_m of class I and rank $r \geq 4$ the equations of Peterson-Codazzi are consequences of the equations of Gauss and that spaces of class p and type $t>2$ are not deformable. The projective properties of rank and type are briefly considered. *M. Knebelman.*

Sen, R. N. On pairs of teleparallelisms. J. Indian Math. Soc. (N.S.) **17**, 21–32 (1953).

Using results of an earlier paper [Bull. Calcutta Math. Soc. **42**, 1–13 (1950); these Rev. **12**, 205] the author obtains conditions that an n-dimensional Riemann space admit certain pairs of parallel displacements. These conditions are then applied to obtain the metric tensors of some three-dimensional Riemann spaces which admit these displacements. Three types of spaces are discussed; the corresponding parallel displacements in two of these types are identified as Clifford's right and left parallelisms and Euclidean parallelism respectively. *A. Fialkow* (Brooklyn, N. Y.).

Kostin, V. I. Some questions of the metric duality of the geometry of Riemann. Trudy Sem. Vektor. Tenzor. Analizu **6**, 444–446 (1948). (Russian)

This is an abstract of a paper that failed to be included in the fifth volume of the above publication. The paper is concerned with the gonometric (angular) invariants of an m-parameter family of hyper-surfaces of an n-dimensional Riemannian space. If $u(x^i, \xi^k)=0$, $i=1, 2, \cdots, n$; $k=1, 2, \cdots, m$, is the equation of the family and $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ is the metric, then the angular metric is $d\theta^2=\Omega_{\gamma\delta}(x, \xi)d\xi^\gamma d\xi^\delta$ where Ω is expressible in terms of $g^{\alpha\beta}$ and the partial derivatives of u with respect to x are ξ.

For two nearby hypersurfaces one must have $u=0$ and $(\partial u/\partial\xi^\alpha)d\xi^\alpha=0$. If, because of these two equations, $\Omega_{\gamma\delta}$ reduce to functions of ξ alone, the angular metric is then Riemannian and is in a sense the dual of ds^2. Writing $u=-x^n+f(x^1, \cdots, x^{n-1}, \xi)=0$, the necessary and sufficient conditions that f must satisfy in order to define a gonometric family is a set of first-order partial differential equations of the second degree, each involving only one pair of parameters. The concluding remarks deal with some integral invariants of the family in case of n-, $(n-1)$- and 1-parameter families. *M. S. Knebelman* (Pullman, Wash.).

Eisenhart, Luther P. Generalized Riemann spaces and general relativity. Proc. Nat. Acad. Sci. U. S. A. **39**, 546–551 (1953).

This paper follows two former publications of the author [same Proc. **37**, 311–315 (1951); **38**, 505–508 (1952); these Rev. **13**, 279; **14**, 317]. There three connexions were considered, $\{{}^h_{ij}\}$ derived from the symmetric part $g_{(ij)}$ of the fundamental tensor, $\Delta^h{}_{ij}$ defined as $\frac{1}{2}g^{(hk)}(\partial_j g_{ak}+\partial_i g_{kj}+\partial_k g_{ij})$ and a third $\Gamma^h{}_{ij}$ defined by $\Gamma^h{}_{(ij)}=\{{}^h_{ij}\}$; $\Gamma^h{}_{[ij]}=\Delta^h{}_{[ij]}$. In this paper another connexion $\Gamma^h{}_{ij}$ is considered, $\Gamma^h{}_{ij}=\Delta^h{}_{ij}+a^h{}_{ij}$, and it is required that $a^h{}_{ij}$ should be chosen in such a way that the covariant derivative $\partial_k g_{ij}-g_{hj}\Gamma^h{}_{ik}-g_{ih}\Gamma^h{}_{kj}$ (note the place of the index k!) be zero. Then it follows that $a^h{}_{[ij]}=-\frac{3}{2}\Delta^h{}_{[ij]}$ gives a possible solution and to this solution belongs $a^h{}_{(ij)}=\frac{1}{2}g^{hk}(g_{[kj]}\Delta^h{}_{[ik]}+g_{[ki]}\Delta^h{}_{[jk]})$. It can be proved that for this solution $\Gamma^k{}_{hk}=0$ and this implies that the divergence of the bivector density $\mathfrak{g}^{ij}g^{[ik]}$ vanishes. From the $\Gamma^h{}_{ij}$ a curvature tensor $\Gamma^h{}_{ijk}$ is derived and this can be put in relation to the curvature tensor of $g_{(ij)}$. It comes out that $\Gamma^h{}_{ijk}$ is just the tensor that Einstein denoted by R_{ij}. This gives relations to Einstein's theory. *J. A. Schouten.*

Schouten, J. A. On subprojective affine connections. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 72–79 (1953).

An A_n is called subprojective if there exists a coordinate system such that every geodesic can be given by $n-2$ homogeneous linear equations and one other equation. A coordinate system of this kind is called special. Following Rachevsky [Trudy Sem. Vektor. Tenzor. Analizu. **1**, 126–142 (1933)] and Kagan [ibid. **1**, 12–101 (1933)], the author obtains the expressions for various geometric quantities including the coefficients of connection and curvature tensors in a special coordinate system. Using these results and an auxiliary algebraic theorem which he derives, the author obtains a more direct proof of the theorem of Adati [Tensor (N.S.) **1**, 116–129 (1951), p. 127; these Rev. **14**, 318] concerning the conditions that an A_n be subprojective. In the course of the discussion, the author shows that one of Adati's conditions is redundant. Two simple solutions of the differential equations occurring in the proof are given. *A. Fialkow* (Brooklyn, N. Y.).

Petrescu, Şt. On the classification of two-dimensional spaces with an affine connection. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 639–646 (1950). (Romanian. Russian and French summaries)

Starting from results of Vranceanu [Leçons de géométrie différentielle, vol. 1, Bucharest, 1947; these Rev. **9**, 532] the author studies two-dimensional affinely connected spaces with a view to classification according to the number of parameters in their maximal group of automorphisms. Canonical forms of space and group are presented. *J. L. Vanderslice* (Silver Spring, Md.).

Vrănceanu, G. On spaces with non-Euclidean affine connection with a maximal group of transformations into itself. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 813–821 (1949). (Romanian. Russian and French summaries)

An affinely connected space which is not euclidean permits at most an n^2-parameter group of automorphisms. The author exhibits the form to which the connection of spaces with maximal group can be reduced as well as the form of the automorphic group itself. *J. L. Vanderslice.*

Vrănceanu, G. On spaces of Kagan. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 299–304 (1950). (Romanian. Russian and French summaries)

This paper provides a classification of the k-fold projective spaces of V. F. Kagan. As a by-product it shows that an $(n-2)$-fold projective space is either subprojective in the sense of Kagan or is the direct product of two flat projective spaces. *J. L. Vanderslice* (Silver Spring, Md.).

Vrănceanu, Gh. On metric spaces of Kagan. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 503–508 (1950). (Romanian. Russian and French summaries)

It is shown that in an n-dimensional Riemann space which is $(n-m-1)$-fold sub-projective (in the sense of Kagan) coordinates can be introduced in such a way that the metric is of constant curvature as regards $n-m$ of them. *J. L. Vanderslice* (Silver Spring, Md.).

Scheibe, Erhard. Über einen verallgemeinerten affinen Zusammenhang. Math. Z. **57**, 65–74 (1952).

This paper is a continuation of G. Lyra's paper on the modification of Riemannian geometry [Math. Z. **54**, 52–64 (1951); these Rev. **12**, 858] and deals with the geometry of affine and projective connections. After discussions of the curvature tensor, the author considers the so-called direction displacement. The author calls two affine connections projectively equivalent, if they have the same direction displacement and he develops the projective theory of the

affine connections in the sense of G. Lyra. [Reviewer's note: It seems that the author does not know the following papers: J. M. Thomas, Trans. Amer. Math. Soc. **28**, 658–670 (1926); V. Hlavatý, Math. Z. **28**, 142–146 (1928); H. Hombu and K. Okada, Proc. Phys.-Math. Soc. Japan (3) **23**, 357–362 (1941) [these Rev. **3**, 20] and the reviewer's paper [Proc. Imp. Acad. Japan **20**, 284–287 (1944); these Rev. **7**, 331] on the transformations of the affine connections of the form $\overline{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta_j{}^i\varphi_k$ and $\overline{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta_j{}^i\varphi_k+\delta_j{}^i\psi_k$.]

K. Yano (Tokyo).

Ruse, H. S. Simply harmonic affine spaces of symmetric connection. Publ. Math. Debrecen **2**, 169–174 (1952).

In a Riemannian space V_n with metric $ds^2=eg_{jk}dx^jdx^k$ ($e=\pm1$), let $P_0(x_0{}^i)$ be a fixed point and $s(x_0, x)$ be the geodesic arc length between two sufficiently near points $P_0(x_0{}^i)$ and $P(x^i)$. Then, if the function $\Omega=\frac{1}{2}es^2$ satisfies $\Delta_2\Omega=f(\Omega)$, V_n is called centroharmonic with respect to P_0 and if this holds for all base points P, it is called completely harmonic. If the function f happens to be a constant, then the space is called simply centroharmonic or simply harmonic according as the solution $\Delta_2\Omega=\text{const.}$ holds for one base point or for all base points.

In the paper under review, the author tries to give definitions of simply centroharmonic and harmonic affine spaces of symmetric connection. The author considers a system of paths with affine parameter t passing through a base point $P_0(x_0{}^i)$ and introduces a vector field by $Y^i=dx^i/dt$. The space is simply centroharmonic with respect to P_0, if $Y^i{}_{;i}=n$ and simply harmonic if this solution holds for all base points. The condition for the space to be simply centroharmonic with respect to P_0 is ${}^*\Gamma^i{}_{jC}y^j=0$ in a normal coordinate system y^i, from which we obtain an infinite sequence of relations.

In the last section, the author discusses the Riemannian extension V_{2n}:

$$ds^2=(c_{ij}-2\Gamma^a{}_{ij}\xi_a)dx^idx^j+2dx^id\xi_i$$

of an affine space A_n of symmetric connection Γ^i_{jk}, where c_{ij} is a given symmetric covariant tensor in A_n. E. M. Patterson [J. London Math. Soc. **27**, 102–107 (1952); these Rev. **13**, 986] has shown that, if V_{2n} is simply harmonic, so is A_n and conversely. In this paper, the author obtains a formula for Ω in V_{2n} and uses it to prove a slightly stronger form of Patterson's theorem, namely that, if V_{2n} is harmonic, then it is simply harmonic and so is A_n; with Patterson's converse that, if A_n is simply harmonic, so is V_{2n}. *K. Yano.*

Ingraham, Richard L. Linear connection in spin-collineation space. Math. Z. **58**, 265–271 (1953).

A q-vector is defined as a contravariant or covariant tensor with q indices which is alternating in all indices. The p-vector covariant derivative of a q-vector over an n-dimensional manifold with a linear connection is defined as the alternating part of the pth covariant derivative of the q-vector. Formal properties of this type of differentiation are given. It is pointed out that a collineation in the spin space, associated with the manifold is determined by ordered collections of p-vectors ($p=0, 1, \cdots, n$) and that the definition of p-vector derivative leads to a definition of a spin-collineation derivative of spin-collineations.

A. H. Taub (Urbana, Ill.).

Ržehina, N. F. The theory of a field of local hypertorses in X_n. Trudy Sem. Vektor. Tenzor. Analizu **9**, 411–430 (1952). (Russian)

This paper is concerned with a one-parameter family of hypertorses (developable hypersurfaces) in a space X_n. The technique is that used in many local problems of differential geometry: with each point of X_n is associated a tangent space E_n with, in this case, a one-parameter family of hyperplanes. Thus one obtains a manifold $X_{n+(1)}$ in which a linear connection is established and by means of it a complete system of differential invariants of the family is obtained. The geometrical guide in this construction is the fact that a one-parameter family of hyperplanes in E_n, not passing through the center, is the dual of a curve in E_n that does not lie in a centro-affine space of lower dimension.

M. S. Knebelman (Pullman, Wash.).

Vagner, V. V. The theory of a field of local curves and of local conical surfaces in X_3 and its applications to the calculus of variations and the theory of partial differential equations. Trudy Sem. Vektor. Tenzor. Analizu **6**, 257–364 (1948). (Russian)

After a discussion of geometrical objects in a one-dimensional space X_1 the theory of curves in a central affine E_3 is developed. Then follows the theory of the cones obtained by connecting points of these curves with the center, which leads directly to the theory of curves in a projective P_3. The transformations $'x^a=x^a/(1-c_\beta x^\beta)$ in the central affine E_3 (they are plane translations $'y_a=y_a-c_a$) then lead to the theory of curves in a central projective space P_3. The next section shows how to introduce a linear connection in the composite $X_{3+(1)}$ obtained by associating to every point of a basic X_3 a local X_1 [see Vagner, C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 94–97 (1943); these Rev. **6**, 106]. Since to every point of an X_3 is associated a central affine E_3, the theory of fields of local curves in an X_3 can now be undertaken; the local curves are taken as the local X_1, so that an $X_{3+(1)}$ arises [see Vagner, ibid. **48**, 229–232, 382–384 (1945); these Rev. **7**, 524, 525]. Then this theory is applied to the simplest case of Lagrange's problem in the calculus of variations [see a.o. G. A. Bliss, Amer. J. Math. **52**, 673–744 (1930)]. The final section deals with the theory of local curves in X_3 given but for local plane translations forming a scalar field ($c_a=\partial_a\sigma$). For the rich details we must refer to the original paper. *D. J. Struik* (Cambridge, Mass.).

Tokarev, P. I. Geometric theory of the second variation in the variational problem of Lagrange. Trudy Sem. Vektor. Tenzor. Analizu **9**, 431–455 (1952). (Russian)

The theory of the variational problems of Lagrange [see Bliss, Amer. J. Math. **52**, 673–744 (1930)] is presented in geometrical form according to the theory of V. V. Vagner [see the paper reviewed above]. The second variations and the equations corresponding to those of Jacobi in the ordinary variation problem are considered in particular. The essential point of Vagner's method is to introduce a local density (for the simple problem) or a local vector (for the general problem) along admissible curves and to apply his theory of local hyperstrip fields and of composite manifolds [same Trudy **8**, 11–72, 197–272 (1950); these Rev. **13**, 281, 778]. In the first of the two parts of this paper, the simple problem of Lagrange, there is defined a given local curve field in X_3: $x^a=l^a(\xi^\lambda, \eta)$, named the indicatrix of the problem, η being a density. Then admissible curves are integrals of the differential equations $m_a{}^{(-1)}\xi^a=0$, $n_a{}^{(-2)}\xi^a=0$, where $m_a{}^{(-1)}$ and $n_a{}^{(-2)}$ represent respectively the components of the affine

binormal and the affine principal normal of the local curve of the indicatrix in a local centro-affine space E_3 associated with the point ξ^λ in X_3. By virtue of the local frame $(l_\alpha, m_\alpha^{(-1)}, n_\alpha^{(-2)})$, the first and second variations of the arc length $s=\int_{t_1}^{t_2} l_\alpha(\xi^\lambda, \eta)\xi^\alpha dt$ and the corresponding equations of Jacobi are expressed in geometrical, invariant forms. The same method is applied also to the Euler-Lagrange multiple rule and similar results are obtained. When the local curve of the indicatrix is a central plane curve, the results have somewhat different forms. In the second part the method is generalized in order to discuss the general problem of Lagrange, by introducing the local $(m-1)$-dimensional hyperstrip field: $x^\alpha=l^\alpha(\xi^\lambda, \eta^a)$, $y_a=l_a(\xi^\lambda, \eta^a)$ $(a, \lambda=1, \cdots, n;$ $a=1, \cdots, m-1)$ in X_n. The system of Pfaff equations $l_{a\xi}{}^\alpha=0$, $n_\alpha{}^p\xi^\alpha=0$ $(p=m+1, \cdots, n)$ defines the basic measurable curves in X_n whose arc lengths are given by $S=\int_0^s l_\alpha(\xi^\lambda, \eta^a)\xi^\alpha ds$. The vectors n^p are defined by $l_a{}^\alpha n_\alpha{}^p=0$, $l^\alpha n_\alpha{}^p=0$, $n_q{}^\alpha n_\alpha{}^p=\delta_q{}^p$, where the vectors $n_p{}^\alpha$ define $(n-m)$-directions, characterizing the hyperplanes of the hyperstrip, i.e., $l_{a\alpha}n^\alpha=0$, $l_\alpha n^\alpha=0$. Then the first and second variations of the integral S and its equations of Jacobi are expressed in terms of the affine invariants, where again Vagner's theory of hyperstrips is applied. Their specific forms are too complicated to be described here.

A. Kawaguchi (Sapporo).

Yano, Shimpei. A generalization of the Frenet's formulae. J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 11–16 (1951).

This is based upon a paper by P. Dienes [Ann. Mat. Pura Appl. (4) **3**, 247–295 (1926)] in which were considered tensors of a very general nature involving position as well as the derivatives of the position coordinates up to a certain order. Monodromic spaces were defined in terms of the form taken by the absolute derivatives of tensors. Isotropic spaces were defined in terms of some further restrictions upon monodromic spaces. The author in this paper points out that both Finsler and Cartan spaces are isotropic spaces in the sense defined, and he proceeds to establish analogues of the Frenet formulae for an arbitrarily given continuous tensor field along a curve. By using the definitions given by Dienes for the angles between tensors he succeeds in giving these Frenet formulae interpretations similar to the corresponding interpretations given in Riemannian geometry.

E. T. Davies (Southampton).

Szmydt, Z. Un théorème de M. Knebelman. Prace Mat.-Fiz. **48**, 101–103 (1952).

Given two Finsler metrics $F(x, p)$ and $\bar{F}(x, p)$, they are said to be conformal if $\bar{g}_{ij}(x, p)=\rho g_{ij}(x, p)$. The reviewer [Proc. Nat. Acad. Sci. U. S. A. **15**, 376–379 (1929)] proved that ρ is necessarily independent of the p's on the assumption that F and $\bar{F}$ are of class C^3. Goląb [Prace Mat.-Fiz. **41**, 97–100 (1934)] showed that class C^2 is sufficient for $n=2$ and the present paper proves that C^2 is sufficient for any n, assuming, however, that $F>0$ for all p's, i.e., that F is definite.

M. S. Knebelman (Pullman, Wash.).

Groenewold, H. J. Spinor rotations and reflections. I, II. Nederl. Akad. Wetensch. Proc. Ser. B. **56**, 169–178, 179–192 (1953).

Spinor representations are given for the groups of rotations and reflections in four- and six-dimensional spaces with indefinite metrics. For various representations the reflection properties of some elementary tensors are collected in a table.

A. H. Taub (Urbana, Ill.).

Kilmister, C. W. A note on Milner's E-numbers. Proc. Roy. Soc. London. Ser. A. **218**, 144–148 (1953).

Recent results of Milner [same Proc. **214**, 292–311, 312–329 (1952); these Rev. **14**, 410] are analysed algebraically. It is shown that Milner's suggested extension of the tensor calculus is unnecessary.

A. H. Taub (Urbana, Ill.).

NUMERICAL AND GRAPHICAL METHODS

***Tables of normal probability functions**

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \quad \text{and} \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\alpha^2}{2}}d\alpha.$$

National Bureau of Standards Applied Mathematics Series No. 23. U. S. Government Printing Office, Washington, D. C., 1953. ix+344 pp. $2.75.

Reprint of a set of tables published first in 1942 [cf. these Rev. **4**, 19] and later issued also in a second edition [1948] as MT14 of the National Bureau of Standards.

***Parrish, W., and Irwin, B. W. Data for X-ray analysis. Vol. I. Charts for solution of Bragg's equation (d versus Θ and 2Θ).** Philips' Technical Library, Eindhoven, 1953. 99 pp. $2.00.

Graphs of the relation $d=\lambda/2 \sin \Theta$ are plotted against both Θ and 2Θ. The values of Θ can be read to .01° and of d to .001°. The several values of λ used in preparing the charts were chosen for their physical interest. [Cf. Tables for conversion of X-ray diffraction angles . . . , National Bureau of Standards Applied Mathematics Series No. 10, Washington, D. C., 1950; these Rev. **12**, 360.]

***Parrish, W., Ekstein, M. G., and Irwin, B. W. Data for X-ray analysis. Vol. II. Tables for computing the lattice constant of cubic crystals.** Philips' Technical Library, Eindhoven, 1953. 81 pp. $2.00.

This volume contains the following tables. For integer N, $1 \leqq N \leqq 378$, all the combinations of integers h, k, l for which $N=h^2+k^2+l^2$. Tables of $\frac{1}{2}\lambda\sqrt{N}$ for such N and for several choices of λ of physical interest. A table of $\sin^2\theta$ for $\theta=10.0°(.1°)89.9°$, 4D [taken from International tables for the determination of crystal structures, vol. II, rev. ed., Edwards Bros., Ann Arbor, Mich., 1944]. A table of $\frac{1}{2}(\cos^2\theta/\sin\theta+\cos^2\theta/\theta)$, $\theta=10.0°(.1°)88.9°$, 3D [reprinted from Nelson and Riley, Proc. Physical Soc. **57**, 160–177 (1945)].

Tikson, Michael. Tabulation of an integral arising in the theory of cooperative phenomena. J. Research Nat. Bur. Standards **50**, 177–178 (1953).

This paper tabulates the integral

$$I\left(\frac{1}{\mu}\right)=\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mu\, dxdydz}{3-\mu(\cos x+\cos y+\cos z)}$$

to five decimals for $\mu=0(.01)1$. For small μ the author obtains

$$I\left(\frac{1}{\mu}\right)=\frac{1}{3}\mu\sum_{m=0}^{\infty}\left(\frac{1}{3}\right)^{2m}(2m)!c_{2m}\mu^{2m}$$

in which the c_{2m} are defined by $[I_0(x)]^3=\sum_0^\infty c_{2m}x^{2m}$, $I_0(x)$ being a Bessel function, and are given to 17 significant figures for $2m=0(2)40$. *J. C. P. Miller.*

Chandrasekhar, S., and Elbert, Donna. The roots of

$$J_{-(l+\frac{1}{2})}(\lambda\eta)J_{l+\frac{1}{2}}(\lambda)-J_{l+\frac{1}{2}}(\lambda\eta)J_{-(l+\frac{1}{2})}(\lambda)=0.$$

Proc. Cambridge Philos. Soc. **49**, 446–448 (1953).

The authors tabulate to 6S the first zero, λ_1, of the equation of the title, together with the values of

$$J_{\pm(l+\frac{1}{2})}(\lambda_1),\quad J_{\pm(l+\frac{1}{2})}(\lambda_1\eta),\quad \frac{2}{\lambda_1^2\pi^2}\left\{\frac{J^2_{l+\frac{1}{2}}(\lambda_1\eta)}{J^2_{l+\frac{1}{2}}(\lambda_1)}-1\right\}$$

for $\eta=.2$, $l=1(1)5$; $\eta=.3$, $l=1(1)6$; $\eta=.4$, $l=1(1)7$; and $\eta=.4, .6, .8$, $l=1(1)15$. *A. Erdélyi* (Pasadena, Calif.).

Stevens, W. L. Tables of the angular transformation. Biometrika **40**, 70–73 (1953).

In the analysis of variance for data expressed as percentages the angular transformation $\theta=50-\lambda \arcsin(1-2p)$, $\lambda=\sqrt{1000}$, or an equivalent transformation is usually made. In table 1 the author gives θ to 3 decimals for $p=0(.001).50$ with proportional parts for linear interpolations. Table 2 gives θ for $p=0(.0001).02$ and Table 3 gives θ for all proper fractions with denominators not exceeding 30, both tables to 3 decimals for θ. *L. A. Aroian.*

Uhler, Horace S. Omnibus checking of the 61-place table of denary logarithms compiled by Peters and Stein, by Callet, and by Parkhurst. Proc. Nat. Acad. Sci. U. S. A. **39**, 533–537 (1953).

This is an account of an extensive program of checking the 61 decimal place common logarithms of Abraham Sharp as reprinted by the editors mentioned in the title. Seven errata were uncovered by adding together logarithms of appropriate primes and applying Stirling's series for $\log\Gamma(x)$. All errata lie beyond the 57th decimal place.

D. H. Lehmer (Los Angeles, Calif.).

von Neumann, J., and Goldstine, H. H. A numerical study of a conjecture of Kummer. Math. Tables and Other Aids to Computation **7**, 133–134 (1953).

Kummer [J. Reine Angew. Math. **32**, 341–359 (1846)] studied the theory of the periods in cyclotomy and was led to consider the counterpart of the Gauss sum in the cubic case, namely

$$x=1+2\sum_{r=1}^{(p-1)/2}\cos(2\pi r^3/p) \tag{1}$$

where p is a prime of the form $6n+1$. This sum and the two others which extend over the two kinds of cubic non-residues modulo p are the three roots of the equation

$$x^3-3px-pA=0, \tag{2}$$

where $4p=A^2+27B^2$, $A\equiv1 \pmod 3$. An unsolved problem is that of deciding in advance for a given p whether the root (1) is the largest, middle or smallest root of (2). Kummer classified the primes p into three classes accordingly and conjectured that the frequencies of these classes are 1/2, 1/3 and 1/6 respectively. His calculations based on the first 45 primes $p=6n+1$ gave densities of .5333, .3111 and .1556.

The present note extends Kummer's calculations to the primes less that 10,000; in all, 611 primes. The results do not bear out Kummer's conjecture. The densities obtained are .4452, .3290 and .2258. These "seem to indicate a trend toward randomness". On the other hand there is room for the conjecture that the ultimate densities are 4/9, 3/9 and 2/9. The calculation was made on the IAS Computer and required about 15 million multiplications.

D. H. Lehmer (Los Angeles, Calif.).

Turing, A. M. Some calculations of the Riemann zeta-function. Proc. London Math. Soc. (3) **3**, 99–117 (1953).

The paper gives an account of two investigations of the zeros of the Riemann Zeta-function in the critical strip. The principal investigation of $\zeta(\sigma+it)$ was for $63^2<t/2\pi<64^2$, that is $24938<t<25736$. This region is at a great distance from the previous calculations of Titchmarsh, which were for $0<t<1468$. A second investigation covered the range $1413<t<1540$. For each of these regions all the zeros of $\zeta(\sigma+it)$ are simple and have $\sigma=1/2$. The larger region contains about 1070 of these zeros. No reason is given for selecting this region. The calculation was performed in 1950 on the Manchester University Mark 1 Computer.

The paper is in two parts. The first part is devoted to the formulation and analysis of the problem along the lines laid down by Titchmarsh [Proc. Roy. Soc. London. Ser. A. **151**, 234–255 (1935)] but with simplifications and certain modifications for machine application. The second part describes the actual calculations as performed on the Manchester machine. The time required to calculate one value of the function in the large interval was about 14 seconds.

Although the author tends to belittle the actual results obtained in a few hours of machine time, the paper shows that a great deal of careful work has been done in preparing the calculation for the machine and this work will be of value to future computers. Since 1950 there has been a large increase in the number and reliability of large scale computers. No doubt further results on this problem will appear in due course. *D. H. Lehmer.*

Bateman, E. H. The solution of algebraic and transcendental equations by iteration. Math. Gaz. **37**, 96–101 (1953).

Let a be an approximation to a root of $f(x)=0$. A refinement of Newton's method for a second approximation, b, is obtained by equating to zero the quadratic as well as linear terms in the Taylor's expansion of $f(a+\lambda)$. The present author linearizes the resulting quadratic equation by replacing one of the λ's in the λ^2-term by the ordinary Newton approximation. In this manner he gets the approximation $\lambda\approx -f/(f'-ff''/2f')$, where f, f', f'' are evaluated at a. It is shown that if $e(a)$ is the error of the approximation a, then $e(b)\approx[e(a)]^3\theta(x)$, where θ depends on derivatives of f up to the third order. A further refinement is obtained by using this last relation with θ calculated at a.

P. W. Ketchum (Urbana, Ill.).

Mikeladze, Š. E. On the theory of construction of interpolation formulas. Doklady Akad. Nauk SSSR (N.S.) **90**, 503–506 (1953). (Russian)

Starting from the Taylor expansion of a function with the remainder in integral form, the author shows how to derive a large variety of interpolation formulas, expressing the desired value of the function in terms of its values and the values of its derivatives at assigned points. One such formula, of interest in the numerical solution of differential

equations, is

$$y(a+h)=y(a)+hy'(a)+\frac{h^2}{2}\Big[y''(a)+\frac{1}{3}\nabla y''(a)+\frac{1}{4}\nabla^2 y''(a)+\frac{19}{90}\Delta^2 y''(a-2h)-\frac{17}{720}\Delta^4 y''(a-2h)+\frac{37}{5040}\Delta^6 y''(a-2h)\Big]-\frac{191}{120960}h^8 y^{(8)}(\xi).$$

W. E. Milne (Corvallis, Ore.).

Stange, K. Über das Ausgleichen von Kurven in Parameterdarstellung. Z. Angew. Math. Mech. **33**, 212–213 (1953).

A graduation problem of classic type is treated by making use of the linear terms in a Taylor expansion. *H. Wold.*

Huber, A. Zur graphischen Berechnung von Polynomen. Z. Angew. Math. Mech. **33**, 248–249 (1953).

Kimball, Bradford F. A multiple group least squares' problem and the significance of the associated orthogonal polynomials. J. Amer. Statist. Assoc. **48**, 320–335 (1953).

Computational procedures are presented to estimate the parameters in the following regression model:

$$Y_{sj}=\alpha_s+\beta_s X+\gamma X^2+\epsilon_{sj},$$

where $s=1, 2, 3$, $j=1, 2, \cdots, N_s$, and ϵ is a random error. There are three groups with a different intercept and slope for each group but a common quadratic term. An orthogonal computing procedure, comparable to the Abbreviated Doolittle procedure, is also presented. A numerical example is included of the use of orthogonal polynomials with a weighted fit. *R. L. Anderson* (Raleigh, N. C.).

Guest, P. G. The Doolittle method and the fitting of polynomials to weighted data. Biometrika **40**, 229–231 (1953).

The Doolittle scheme of solving normal equations is extended to the case of weighted observations, using both power moments and factorial moments for the polynomials. *R. L. Anderson* (Raleigh, N. C.).

Piotrowski, S. Ludwik. The problem of weights in the least squares solution of non-linear equations. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **43** (1950), 4–12 (1952).

In the case of non-linear equations the standard formulas of the method of differential corrections in the least squares solution do not lead exactly to the weights of the unknowns, but to quantities which approximate the weights if the errors are small. The author discusses this situation in some detail and gives a geometric interpretation. *W. E. Milne* (Corvallis, Ore.).

Bjerhammar, Arne. Application of calculus of matrices to method of least squares with special reference to geodetic calculations. Trans. Roy. Inst. Tech. Stockholm no. **49**, 86 pp. (2 plates) (1951).

This is a brief handbook on the calculational aspects of the adjustment of observations by the method of least squares. Different types of procedures are given a uniform treatment as follows. The functional relations are expressed in a rectangular matrix equation. The general solution of the latter is obtained by premultiplication with the author's "rectangular reciprocal matrix" [see Bull. Géodésique **1951**, 188–220 (1951); these Rev. **13**, 312]. The conditions are used to select the unique solution from the general solution.

A special study is made of the various forms of adjustment obtained by partitioning the rectangular matrix. The numerical methods of solving the rectangular equation are reviewed; there is a bibliography of 43 titles. A new method of solving normal equations is deduced. The procedure is illustrated numerically with a geodetic system leading to normal equations of order 28.

There is a resumé of the properties of rectangular reciprocal matrices, including the result on the characteristic polynomial and rank of AA^{-1} rediscovered by Stojaković [C. R. Acad. Sci. Paris **236**, 877–879 (1953); these Rev. **14**, 1055]. *G. E. Forsythe* (Los Angeles, Calif.).

Karush, W. Convergence of a method of solving linear problems. Proc. Amer. Math. Soc. **3**, 839–851 (1952).

A is a bounded self-adjoint linear operator on a real Hilbert space $\mathfrak{C}$ which may be separable without loss of generality. Lanczos [J. Research Nat. Bur. Standards **45**, 255–282 (1950); these Rev. **13**, 163] has described an iterative method for solving two problems associated with A: (1) finding λ and y such that $Ay=\lambda y$; (2) finding x such that $(A-\lambda I)x=b$. Karush here formulates the method as follows: Let b be arbitrary for (1), or given for (2). Let $\mathfrak{C}_i$ be the i-space spanned by $b, Ab, \cdots, A^{i-1}b$. Project the operator on $\mathfrak{C}_i$, solve the corresponding problem (1_i) or (2_i) in $\mathfrak{C}_i$, and let $i\to\infty$.

The following are proved: (I) Under certain assumptions on the spectrum of A, the characteristic vectors and values of problem (1_i) converge to those of (1), as $i\to\infty$, the values monotonically. (II) If A is completely continuous the convergence in (I) is faster than that of any geometric progression. (III) If A is completely continuous and $\lambda\neq 0$ is not a characteristic value of A, then the convergence of the solution of problem (2_i) to that of (2) is faster than that of any geometric progression. *G. E. Forsythe.*

Lopšic, A. M. An extremal theorem for a hyperellipsoid and its application to the solution of a system of linear algebraic equations. Trudy Sem. Vektor. Tenzor. Analizu **9**, 183–197 (1952). (Russian)

Let A be a symmetric positive definite affinor [=matrix] whose largest and least proper values are M and m. Let ϕ be the angle made by the radius vector r at the point P of the hyperellipsoid $rAr=1$ with the "principal normal plane" at P—i.e., with the subspace spanned by the vectors Ar, $A^2r, \cdots, A^kr$. Let $T_k(x)$ be the Čebyšev polynomial, with $T_k(1)=1$. Theorem: Always $\sin\phi\leqq[T_k(\delta)]^{-1}$, where $\delta=(M+m)(M-m)^{-1}$. The proof, which is expressed in terms of multivectors, is long.

The theorem is used to estimate the convergence of a proposed class of gradient methods of solving a linear system $Ax=a$: Let $x_0=0$, $a_0=a$. Given x_i and $a_i=a-Ax_i$, let $x_{i+1}-x_i$ be so chosen in the k_i-dimensional ($k_i\geqq 1$) space Π_i spanned by $a_i, Aa_i, \cdots, A^{k_i-1}a_i$ that $|Ax_{i+1}-a|$ is minimized. It is shown that $|a_{i+1}|\leqq|a_i|\sin\phi_i$, where ϕ_i is the angle made by a_i with the plane $A\Pi_i$.

For $k_i=1$ the author's method is the "1-process" described also by Krasnosel'skiĭ and Kreĭn [Mat. Sbornik N.S. **31(73)**, 315–334 (1952); these Rev. **14**, 692, q. v.]. The author states that the 1-process requires one multiplication of A by a vector per step, whereas the related "0-process" of Kantorovič and others takes two. [The reviewer can find no such difference.]

The reviewer suspects the theorem could be proved briefly and elegantly following the Birman paper cited by the author [Uspehi Matem. Nauk (N.S.) 5, no. 3(37), 152–155 (1950); these Rev. 12, 32; 14, 412].

G. E. Forsythe (Los Angeles, Calif.).

Rosser, J. Barkley. A method of computing exact inverses of matrices with integer coefficients. J. Research Nat. Bur. Standards 49, 349–358 (1952).

The exact inverse of a matrix A with small integer coefficients has rational coefficients with numerators and denominators which are ordinarily exceedingly large. If A^{-1} is to be computed by machine, it is highly desirable to postpone as long as possible the growth of the numbers beyond the capacity of the registers. In ordinary exact elimination by row-multiplication and row-subtraction, the numbers grow rapidly, even where one takes the time to (*): remove common factors when combining equations.

The author proposes changing A to a diagonal matrix by a succession of row-subtractions (or additions), with a few interchanges of rows and no row-multiplications. While working on the kth column, at each stage one operates with the two rows whose coefficients in the kth column numerically dominate the other coefficients of the column. A^{-1} is obtained by performing the same row operations on I. The procedure is said to be easily mechanizable for punched-card machines, and to have the desired effect of postponing the growth of the numbers.

The proposed procedure and ordinary elimination with refinement (*) were both carried out on a desk computer for a certain matrix of order 6 with $\max|a_{ij}| = 195$. The times, including training of the operator, were 23 hours and 38 hours, respectively. Many of the calculations are reproduced in the paper. *G. E. Forsythe* (Los Angeles, Calif.).

Burger, A. P. Inversion of matrices with the aid of punched card machines. Statistica, Rijswijk 6, 121–133 (1952). (Afrikaans. English summary)

Details are given for the inversion with Hollerith punch-card equipment of symmetric matrices A by the Cholesky method of finding a lower triangular matrix P such that $PP' = A$. The machines used were a hand-punch, Senior totalroll tabulator (quivalent to a tabulator with a cardcycle total transfer device), a "usual" 10-place multiplier, a reproducer with summary punch, and a sorter. Accuracy of results, number of operations, time required, and operating procedures are discussed. A matrix of order 48 was inverted in the South African Weather Bureau in less than two months. *G. E. Forsythe* (Los Angeles, Calif.).

Kac, A. M., and Čekmarev, A. I. On the computation of Hurwitz determinants. Akad. Nauk SSSR. Inženernyĭ Sbornik 12, 221–222 (1952). (Russian)

A simple elimination scheme is presented for the evaluation of Hurwitz determinants used in calculating the stability of systems of automatic regulation. The determinants are nearly triangular, and the scheme makes them triangular.

G. E. Forsythe (Los Angeles, Calif.).

Lavut, A. P. On the disposition of the characteristic numbers of Seidel's transformations for systems of normal equations. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 197–202 (1952). (Russian)

Seidel's method of solving the linear system $Ax+b=0$ of order n, where A is positive definite, always converges. Let A_s be the matrix corresponding to one cycle of the Seidel process. It follows that all proper values of A_s are less than one. The following result is proved: given $\epsilon>0$ and any complex λ_0, $|\lambda_0|<1$, one can find an n and an A such that one of the proper values of A_s is closer than ϵ to λ_0.

G. E. Forsythe (Los Angeles, Calif.).

Bodewig, E. Bericht über die Methoden zur numerischen Lösung von algebraischen Eigenwertproblemen. II. Atti Sem. Mat. Fis. Univ. Modena 5, 3–39 (1951).

This is the second half (direct methods and 36 references) of a valuable report intended to compare all known numerical methods for finding eigenvalues and eigenvectors of finite matrices A. [For the first half, see the same Atti 4, 133–193 (1950); these Rev. 13, 991.]

Ordinarily the direct methods solve the problem by: (1) determining the characteristic polynomial $D(t)$ of A; and (2) finding the desired zeros t_i of D, which are eigenvalues of A. To get the eigenvector x_i belonging to a t_i, the author recommends that one: (3) get

$$P_i(t) = D(t)/(t-t_i) = t^{n-1} + b_1 t^{n-2} + \cdots + b_{n-1},$$

choose almost any vector v, and then compute

$$x_i = P_i(A)v = A^{n-1}v + b_1 A^{n-2}v + \cdots + b_{n-1}v.$$

The alternative to (3) of solving the linear system $(A-t_iE)x_i=0$ (E=identity) is ordinarily too long. The author discusses the necessary modifications of (1) and (3) when $D(t)$ has multiple zeros.

There are brief expositions and counts of operations for the following methods of obtaining $D(t)$: (i) the Leverrier-Horst method of putting the trace of the powers $A, \cdots, A^n$ into Newton's identities, and a modification due to Souriau (independently proposed by Frame and by Faddeeva and Sominskiĭ); (ii) the Krylov-Lusin-Duncan method of determining the linear dependence between $v, Av, \cdots, A^nv$; (iii) the Hessenberg method of using simple similarity transformations to put A into an almost triangular form; (iv) Danilewski's method of bringing $A-tE$ into the Frobenius rational canonical form by elementary transformations; (v) Weber's method of bringing $A-tE$ into a certain triangular form; (vi) Samuelson's method of condensing the system of differential equations $Ax^{(0)}=x^{(i+1)}$ ($i=0, 1, \cdots, n-1$) into a single system of order n; (vii) the method of calculating $D(t)=\det(A-tE)$ for $t=0, 1, \cdots, n-1$, and then solving a linear system for the coefficients of $D(t)$. The reviewer noted no omissions of direct methods known then, but now one should add that of C. Lanczos [J. Research Nat. Bur. Standards 45, 255–282 (1950); these Rev. 13, 163].

There is also a discussion of solving the problem for neighboring systems $A+dA$, knowing the solution for A. There are some historical remarks, especially on the Leverrier-Jacobi rivalry, and various criticisms of the literature.

In a summary the author gives a detailed outline of methods recommended for various goals, noting that the choice may depend also on the available computing aids. As a standard overall procedure he recommends method (ii). To the extent that one eigenvalue t_1 dominates the others, it and the corresponding vector x_1 will already be available from $A^{n-1}v$ and A^nv. In general, the author much prefers direct methods, because they require fewer arithmetic operations. Iterative methods are recommended when one seeks only the dominant eigenvalue (ordinary iteration on a vector) or when A is symmetric and has a strongly dominating diagonal (Jacobi method). *G. E. Forsythe.*

Herrmann, A. Bestimmung der höheren Eigenwerte einer Matrix durch Iteration. Ann. Univ. Saraviensis 1, 220–223 (1952).

Assume the matrix A to have only real eigenvalues $\lambda_1, \cdots, \lambda_n$. The author proposes an adaptation of ordinary vector iteration which will converge to the eigenvector y_0 belonging to the eigenvalue nearest to a preassigned value λ_0. The idea is to form the polynomial $f(\lambda)=\lambda''-(\lambda-\lambda_0)^2$, for a suitably chosen λ''. For the iteration $z^{(k+1)}=f(A)z^{(k)}$, one has $z^{(k)}\to y_0$. The iterates $z^{(k)}$ can of course be formed without squaring a matrix. *G. E. Forsythe* (Los Angeles, Calif.).

Mayanc, L. S. A method for making more precise the roots of secular equations of high degrees and for numerical analysis of their dependence upon parameters of the corresponding matrices. Doklady Akad. Nauk SSSR (N.S.) 50, 121–124 (1945). (Russian)

Let W be a matrix of order n. The problem is to design a rapidly convergent iterative scheme for computing an eigenvalue λ and corresponding eigenvector r of W, given some initial approximations λ_0, r_0. The approach considered here is the iterative solution of the non-linear system of equations (*) $(W-\lambda E)r=0$ (E = unit matrix), subject to the restriction $e_1{}^T r=1$, $e_1{}^T=(1, 0, \cdots, 0)$. Earlier algorithms by the author and by Hopšteln did not always converge [they were of the cyclic single-step category], but one is now proposed which is claimed to work always, at the cost of one matrix inversion per step.

In the reviewer's notation the author assumes $w_{11}=0$ without loss of generality, by subtracting $w_{11}E$ from W. He then partitions W into submatrices of orders 1 and $n-1$:

$$W=\begin{bmatrix} 0 & w_r \\ w_e & B \end{bmatrix}$$

and writes $r=[1, x]^T$. Then, given λ_k, x_k, he gets λ_{k+1}, x_{k+1} by the formulas

$$\begin{cases} \lambda_{k+1}=\delta^{-1}[w_r(B-\lambda_k E)^{-1}x_k\lambda_k - w_r(B-\lambda_k E)^{-1}w_e], \\ \quad \text{where} \quad \delta=w_r(B-\lambda_k E)^{-1}x_k-1, \\ x_{k+1}=(B-\lambda_{k+1}E)^{-1}w_e. \end{cases}$$

If the first approximation is good enough, some saving of effort results from the alternate formula

$$\lambda_{k+1}=\delta_0^{-1}[w_r(B-\lambda_0 E)^{-1}x_0\lambda_k - w_r(B-\lambda_k E)^{-1}w_e],$$

where

$$\delta_0=w_r(B-\lambda_0 E)^{-1}x_0-1.$$

There are related formulas for computing the derivatives of λ with respect to a parameter on which W depends. There are no theorems or proofs in the paper.

Reviewer's note: the first formula for λ_{k+1} corresponds to one step of Newton's process for solving the system (*) in the n-dimensional space of λ and the last $n-1$ components of r. *G. E. Forsythe* (Los Angeles, Calif.).

Svirskiĭ, I. V. On estimation of the accuracy of variational methods of determining characteristic values. Doklady Akad. Nauk SSSR (N.S.) 87, 889–892 (1952). (Russian)

Let H be a self-adjoint operator, defined over $D(H)$, bounded from below, whose characteristic values λ_k we seek to estimate. The Timoshenko-Ritz method gives upper bounds $\lambda_{k\delta}$ for all λ_k, and to bound the error $\lambda_{k\delta}-\lambda_k$ one needs lower bounds for λ_k. Definition: $H_0 \leqq H$ with respect to a function-class M if $(H_0 u, u) \leqq (Hu, u)$ for all $u \in M$. If $M=D(H)$, then any H_0 leads by the "method of comparison" to a lower bound for each λ_k. The author gives two methods of successive approximation which, starting from $H_0 \leqq H$, determine an $H_4 \leqq H$ which gives close lower bounds for λ_k. In each method it is necessary to know the characteristic values of H_0. In one method it is also necessary to know the characteristic functions of H_0. *G. E. Forsythe.*

Svirskiĭ, I. V. On the accuracy of Galerkin's method. Doklady Akad. Nauk SSSR (N.S.) 88, 757–760 (1953). (Russian)

The author obtains estimates for the error of Galerkin's method of solving an equation $Hf(x)=h_1(x)$, where H is a symmetric linear differential operator, bounded from below, with a point spectrum. For definiteness the author confines his attention to an equation for the forced vibration of a string: $Hf \equiv -\partial_x{}^2 f(x)-(\rho\omega^2/T)f(x)=h_1(x)$, subject to $f(0)=f(l)=0$. It is shown that, for $0<a<l$,

$$f(a)=(H^{-1}h_1, h_2)-(h_1, u_1),$$

where $h_2=Hu_1$, where $u_1=u_1(a, x)$ is Green's function for the problem, and where $(f, g)=\int_0^l fg\,dx$. It is next shown that, for all f, $([H_n]^{-1}f, f) \leqq (H^{-1}f, f) \leqq (H_4^{-1}f, f)$, where $[H_n]^{-1}f$ is the Galerkin solution of the problem and H_4 is the operator obtained in an earlier paper [see the preceding review]. Out of the above, if $[H_n]^{-1}$ and H_4 have the same number of negative characteristic values, a two-sided bound is obtained for $(H^{-1}h_1, h_2)$ in terms of the quantities $([H_n]^{-1}h_i, h_j)$ and $(H_4^{-1}h_i, h_j)$ $(i, j=1, 2)$. This yields two-sided bound for $f(a)$. *G. E. Forsythe.*

v. Sanden, H. Differentiation durch Integration. Z. Angew. Math. Mech. 33, 250 (1953).

Grossman, D. P. On formulas of numerical differentiation. Doklady Akad. Nauk SSSR (N.S.) 89, 777–779 (1953). (Russian)

The author obtains a formula for the kth derivative of a function $f(x)$ at a point a, expressed in terms of functional values at arbitrarily given points. Two forms of the remainder are presented, one for the case where $f(x)$ is analytic in a circle with center at a in the complex x-plane and such that the given points lie in the circle; the other valid for the case of x real, with the assumption that $f(x)$ has a derivative of order equal to or greater than the number of given points. *W. E. Milne* (Corvallis, Ore.).

Tollmien, W. Bemerkung zur Fehlerabschätzung beim Adamsschen Interpolationsverfahren. Z. Angew. Math. Mech. 33, 151–155 (1953).

This paper (dedicated to R. v. Mises for his seventieth birthday) is devoted to a detailed discussion of the error in Adams' method, a topic that had been studied earlier both by v. Mises and by Tollmien. The author examines and explains a difficulty encountered in connection with his earlier expression for the error in Adams' method. This difficulty lay in the fact that the desired function is represented in successive intervals by different polynomials.

W. E. Milne (Corvallis, Ore.).

Azbelev, N. V. On the limits of applicability of S. A. Čaplygin's theorem. Doklady Akad. Nauk SSSR (N.S.) 89, 589–591 (1953). (Russian)

The absence of estimates on the limits of applicability of the theorem on differential inequalities impedes the development of Čaplygin's method for approximate integration of differential equations. To supply this want the author

establishes a comparison theorem for the solution of a differential system consisting of an nth order ordinary differential equation with assigned initial values for the unknown and its first $n-1$ derivatives. The underlying idea is to set up a dominating differential system which is linear.

W. E. Milne (Corvallis, Ore.).

Roberson, R. E. On an iterative method for nonlinear vibrations. J. Appl. Mech. **20**, 237–240 (1953).

This is essentially Gwoh-Fan Djang's modification of Picard's method of successive approximations, namely, that in which the new approximation for each variable is used in the iteration as soon as it is obtained. The author applies the method to dynamical systems with one and two degrees of freedom, and to dissipative system. *W. E. Milne.*

Ladyženskaya, O. A. On application of the method of finite differences to the solution of Cauchy's problem for hyperbolic systems. Doklady Akad. Nauk SSSR (N.S.) **88**, 607–610 (1953). (Russian)

This summarizes results obtained in the author's dissertation on numerical solution of systems

$$\frac{\partial u_i}{\partial t}=\sum_{j=1}^{N}\sum_{k=1}^{m}a^k_{ij}(t,x,u)\frac{\partial u_j}{\partial x_k}+b_i(t,x,u)$$

with initial conditions $u_i=\phi_i(x)$, where the arguments x and u denote $x_1,\cdots,x_m$ and $u_1,\cdots,u_N$, and ϕ_i are periodic with period l in each x. If $\partial u_i/\partial t$ are approximated by $[u(t+\Delta t,x)-u(t,x)]/\Delta t$, and $\partial u/\partial x_k$ by

$$[u(t+\Delta t,\cdots,x_k+\Delta x_k,\cdots,x_m)-u(t+\Delta t,\cdots,x_k,\cdots,x_m)+u(t,\cdots,x_k+\Delta x_k,\cdots,x_m)-u(t,\cdots,x_k,\cdots,x_m)]/4\Delta x_k$$

with $\Delta x_k=l/n$, the corresponding system of difference equations is satisfied by trigonometric polynomials $u_{i\Delta}$. The author asserts, without specifying hypotheses on a^k_{ij}, b_i, and ϕ_i, that as Δt and Δx tend to zero $u_{i\Delta}$ converge uniformly to the solutions of the partial differential equation when $\Delta t/\Delta x$ is less than some constant. Other approximations to $\partial u_i/\partial x_k$ by means of central, forward, or backward differences with respect to x_k at time t are said to lead in general to divergent processes. *J. H. Giese.*

***Levy, Samuel, and Kroll, Wilhelmina D. Errors introduced by finite space and time increments in dynamic response computation.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 1–8. The American Society of Mechanical Engineers, New York, N. Y., 1952.

A number of vibration problems, involving a simple uniform beam or two beams forming a cross, were calculated by lumping parameters and integrating numerically. Results are grouped to show the effect of choosing (a) different time increments and (b) different methods of lumping the mass. The authors find that the approximation formula

$$\ddot{x}_n\doteq(\Delta t)^{-2}(2x_n-5x_{n-1}+4x_{n-2}-x_{n-3})$$

yields more stable results than the usual

$$\ddot{x}_n\doteq(\Delta t)^{-2}(x_{n+1}-2x_n+x_{n-1}).$$

They conclude that a time increment of 1/30 the fundamental period of the system, and replacement of the simple beam by a five mass idealization, gives good results.

R. E. Gaskell (Seattle, Wash.).

Cowling, T. G. A new method of numerical integration of the equations of the laminar boundary layer. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2575 (8690), 14 pp. (1951).

Goodell, John D. Notes on decision element systems using various practical techniques. J. Computing Systems 1, 196–199 (1953).

Angelo, E. J., Jr. An electron-beam tube for analog multiplication. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. No. 249, i+41 pp. (1952).

Belgrano Brémard, J. C. Sur un problème de calcul fonctionnel posé par la nomographie. Revista Mat. Hisp.-Amer. (4) **13**, 81–86 (1953).

The nomogram considered relates four variables z_i. Its nature and method of employment is as follows: Through point P labeled (z_1, z_2) on a grid draw a line of fixed direction meeting in point R the curve labeled z_4 of a family. Join point Q labeled z_3 on a scale to fixed point O. Then $PR=OQ$. A functional relation can be represented by such a nomogram if it can be reduced to the form $z_4=F(f_{12}+g_3, h_{12})$ in the usual notation. The condition that such reduction be possible is the satisfaction of two non-linear partial differential equations of the first order in two unknowns.

J. M. Thomas (Durham, N. C.).

Belgrano Brémard, J. C. On a problem of functional calculus posed by nomography. Revista Mat. Hisp.-Amer. (4) **13**, 87–91 (1953). (Spanish)

Spanish version of the paper reviewed above.

***van Veen, H. J. Inleiding tot de nomographie.** [Introduction to nomography.] 2d printing. P. Noordhoff N. V., Groningen-Djakarta, 1952. 197 pp. fl. 12.50.

This book, whose present second printing differs only slightly from the first (1937), is an introductory text adequate to give a working knowledge of alignment charts and allied nomograms. The exercises are plentiful and substantial.

J. M. Thomas (Durham, N. C.).

Jeger, M. Topologische Gesichtspunkte in der Nomographie. Eine Einführung in die Geometrie der Gewebe. Elemente der Math. **8**, 25–31, 49–53 (1953).

In this expository account a family of curves is defined as the topological image of a family of parallel chords of a curve which is met in just two points by the support of each of those chords. A web is composed of three families whose curves have tangents, the three tangents at each point being distinct. A web of straight lines and an alignment nomogram are equivalent under a correlation. The disjunction problem for an equation is given the form: given an equation $F(x,y,z)=0$, do the curves $x=$const., $y=$const., $z=$const. constitute a web topologically equivalent to a web of straight lines? It is pointed out that the Blaschke-Thomsen theorem answers the question when the straight lines in each family are parallel.

J. M. Thomas (Durham, N. C.).

Watanabe, Sigekatu. On nomography. Rep. Univ. Electro-Commun. **1950**, no. 2, 251–267 (1950). (Japanese)

The author's standpoint is as follows: If we try to realize on a sheet of paper the axiomatic point algebra stated in Veblen and Young's Projective geometry [2 vols., Ginn, Boston-New York, 1910, 1919], then this point algebra can be regarded as a kind of nomography. *K. Yano.*

ASTRONOMY

Agostinelli, Cataldo. Figure di equilibrio prossime all'ellissoide di una massa liquida omogenea attratta da più corpi lontani con la legge di Newton. Rend. Circ. Mat. Palermo (2) 1 (1952), 281–322 (1953).

The existence of figures of a liquid mass other than ellipsoids is investigated under conditions mentioned in the title of the paper. The author proved in an earlier investigation [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (2) 71, 1–48 (1951); these Rev. 14, 94] that an invariable ellipsoid can be the free surface, the paths of internal movements being elliptical. Now, it is found that there is a set of critical ellipsoids such that an infinitesimal deformation of them can produce new figures of equilibrium. Obviously this conclusion holds as long as the motion of centers of attraction is neglected. *W. Jardetzky* (New York, N. Y.).

Bondi, H., and Lyttleton, R. A. On the dynamical theory of the rotation of the earth. II. The effect of precession on the motion of the liquid core. Proc. Cambridge Philos. Soc. 49, 498–515 (1953).

In this part of their paper [for part I see same Proc. 44, 345–359 (1948); these Rev. 10, 160] the authors attempt to determine the motion of a viscous liquid core of a planet on linearizing the general equations. The investigation suggests that the internal movements affected by the precession of a rigid shell cannot correspond to a steady state. Despite the simplifications (spherical cavity, uniform density of an incompressible liquid, very small angular velocity of precession) the linearized equations for the small departure from the rigid body rotation remain extremely complicated. On discussing the motion in the boundary layer and also in a deeper internal region of the core, Bondi and Lyttleton show some difficulties connected with the boundary conditions. They derive a particular form of solution for the boundary layer representing the changes in velocity components across the depth of the layer. Doubt is voiced by the authors concerning the possibility of an internal motion which could be represented everywhere by analytic functions. It seems that the precessional motion of the Earth should result in turbulence in the core. The main turbulence (in latitudes 30°N and 30°S) would introduce couples acting on the shell.

The approach in this paper does not follow the lines of some other investigations concerning the problem. See, for example, the discussion in Stekloff, Ann. Fac. Sci. Univ. Toulouse (3) 1, 145–256 (1909), a paper which is also not mentioned in Lamb's Hydrodynamics, 6th ed., Cambridge, 1932. *W. S. Jardetzky* (New York, N. Y.).

Pilowski, K. Bemerkungen zu den Grundlagen der sphärischen Astronomie. Astr. Nachr. 281, 59–67 (1953).

The proposal is made that the spherical coordinate system of a fundamental star catalogue be defined by two suitably chosen stars with small proper motions. The absolute proper motions in such a system could eventually be obtained by referring the stars to a background of distant extragalactic nebulae. The object is to achieve a separation of the astronomical problem of determining star positions in a practically fixed coordinate system from the more geodetic and geophysical problems that concern the motions of the earth. Time determination and the use of meridian circle observations for obtaining positions in the proposed coordinate system require a method of reduction different from the established practice. Some of these questions are treated in the later sections of the article. *D. Brouwer.*

***Herrick, Samuel. Tables for rocket and comet orbits.** National Bureau of Standards Applied Mathematics Series, No. 20, U. S. Government Printing Office, Washington, D. C., 1953. xxiv+100 pp. $1.75.

The anticipated development of rocket navigation has directed serious attention, for the first time, to rectilinear motion in the two-body problem. The accompanying tables of $\sin E$ and $1-\cos E$ with argument $E-\sin E$ and the parallel tables in the hyperbolic functions make it possible, for rectilinear orbits, to determine position and velocity from the time. This is accomplished, moreover, by direct interpolation without the aid of series expansions or successive approximations. The inverse process, the determination of the time from position or velocity, is solved by inverse interpolation; the tables were designed primarily for the first of these processes, the more common of the two.

The tables may be used also in connection with nearly rectilinear motion, that is, motion in ellipses and hyperbolas whose eccentricities are near unity. Orbits of these types, cometary orbits, have generally been treated as "nearly parabolic", that is to say, they have been handled by special tables, based upon series expansions, that correct the position and velocity that would be obtained for a parabola having the same perihelion distance. The "worm's eye view" resemblance of these orbits to the parabola near perihelion is illusory; actually they are more closely and simply related to rectilinear orbits of the same major axis.

Astronomers will be tempted, upon first consideration, to suspect that the tables will not work near perihelion for "nearly parabolic" orbits. Surprisingly enough, they do. The critical test is not the "nearly parabolic" orbit but the rectilinear one; for both kinds of orbits the device that makes the table usable is its subdivision into "ranges", the first of which, designated by "$n \geqq 5$", may be used over and over again like a logarithm table or a table of powers (which it is), and yields at least eight significant figures, no matter how closely the moving object may approach to perihelion, perigee, or coincidence. Actually, these tables provide a treatment that is more general than the "nearly parabolic" one; it will handle the parabola, whereas the latter cannot handle the rectilinear orbit. The functions tabulated, moreover, have more general interest and broader application than those of the nearly parabolic tables. (From author's introduction.)

Contents. Introduction. 1. Rectilinear motion; origin of the functions. 2. Direct interpolation (examples, rectilinear motion). 3. Inverse interpolation (examples, rectilinear motion). 4. Nearly rectilinear or "nearly parabolic" motion. 5. Position and velocity from time (examples, parabolic and nearly rectilinear motion). 6. Time from position and velocity (examples, nearly rectilinear motion). 7. Method of computation of the tables. 8. Acknowledgements. 9. References (17 items). Schedule A: Ranges of values for various n's. Tables: Parabola: $\frac{1}{6}D^3$ and D with argument $\frac{1}{6}D^3 = 10^{-3n}u$, all values of n. Ellipse: $1-\cos E$, E with argument $E-\sin E = 10^{-3n}u$. Hyperbola: $\cosh F - 1$, $\sinh F$, F with argument $\sinh F - F = 10^{-3n}u$ for $n \geqq 5$, $n = 4, 3, 2, 1, 0$. (E is the eccentric anomaly for the ellipse; F is the corre-

sponding angle for the hyperbola; D is a new auxiliary for the parabola.) Ev t interpolation coefficients.

E. Leimanis (Vancouver, B. C.).

Fleckenstein, J. O. Les cas d'exception au théorème de Laplace sur les perturbations séculaires des éléments vectoriels des orbites planétaires. Experientia **9**, 252 (1953).

Soudan, Adel. Sur de nouveaux éléments canoniques du mouvement elliptique. C. R. Acad. Sci. Paris **236**, 1533–1535 (1953).

The solution of Jacobi's partial differential equation for the two-body problem is written in the generalized form:

$$S=\int_{r_0}^{r}\left(2h+\frac{2\mu}{r}-\frac{(G-F)^2}{r^2}\right)^{\frac{1}{2}}dr+\Theta\psi+\int_{0}^{\omega}\left(G^2-\frac{\Theta^2}{\cos^2\omega}\right)^{\frac{1}{2}}d\omega,$$

which differs from the usual form by the replacement of G by $G-F$ under the first radical. The availability of $\rho=F/(G-F)$ as a constant or a variable makes it possible to obtain the systems of canonical variables obtained by Delaunay, Levi-Cività, Hill, de Sitter and Jekhowsky as special cases of a single more general form. In addition, a new system is introduced by the author. The analysis is closely related to that given by H. Andoyer [Bull. Astr. **30**, 425–429 (1913)]. *D. Brouwer* (New Haven, Conn.).

Meffroy, Jean. Sur les termes séculaires du développement des grands axes par rapport aux masses. II. C. R. Acad. Sci. Paris **236**, 2482–2484 (1953).

This is a continuation of an earlier note by the same author [same C. R. **236**, 778–780 (1953); these Rev. **14**, 802]. For the case that the eccentricities and inclinations are zero, an expansion of the disturbing function with the aid of Legendre polynomials is used. The expression for the secular part of $\delta^3 a$, the perturbation of the third power of the masses in the semi-major axis, is obtained as a series in powers of $(a/a')^4$ beginning with the term $(a/a')^6$. It is concluded that if the eccentricities, inclinations, and the ratio a/a' are sufficiently small and arbitrary, $\delta^3 a$ contains a purely secular term. *D. Brouwer.*

Grémillard, Jean. Sur la recherche des solutions périodiques de la troisième sorte. C. R. Acad. Sci. Paris **236**, 1952–1953 (1953).

In order to find the periodic solutions of the third kind in the three body problem by the method of Poincaré [Les méthodes nouvelles de la mécanique céleste, vol. 1, Gauthier-Villars, Paris, 1892, pp. 144–152], one has to determine the values of the eccentricities and the inclinations of the orbits for which the mean value of the perturbative function R assumes a maximum or a minimum. Moreover, the two eccentricities and the two inclinations are related by the two integrals

$$f_1\equiv\lambda(1-e^2)^{\frac{1}{2}}\cos i+\lambda'(1-e'^2)^{\frac{1}{2}}\cos i'-C=0,$$
$$f_2\equiv\lambda(1-e^2)^{\frac{1}{2}}\sin i+\lambda'(1-e'^2)^{\frac{1}{2}}\sin i'=0,$$

where $C\neq 0$ denotes the constant of angular momentum. As is well known, the problem is equivalent to that of finding the extreme values of the function $R+k_1f_1+k_2f_2$, containing two Lagrangian multipliers k_1 and k_2. The author shows that one of the two Lagrangian multipliers is zero, thus justifying Poincaré's formulas, which involve only one multiplier.

E. Leimanis (Vancouver, B. C.).

Keller, Geoffrey. Astronomical "seeing" and its relation to atmospheric turbulence. Astr. J. **58**, 113–125 (1953).

A method is outlined by which it is, in principle, possible to compute the average autocorrelation function of density fluctuations in the earth's atmosphere, assuming that the distribution of light intensity in the "seeing" image of a star is known. The presentation is in two parts, the first being devoted to a qualitative description of the events which occur. The second part contains the mathematical theory. *Author's summary.*

RELATIVITY

Hlavatý, V., and Sáenz, A. W. Uniqueness theorems in the unified theory of relativity. J. Rational Mech. Anal. **2**, 523–536 (1953).

This paper, which is closely related to an earlier one by Hlavatý [same J. **2**, 1–52 (1953); these Rev. **14**, 505], deals with the problem of finding necessary and sufficient conditions that the equations

$$(*)\qquad \partial_\alpha g_{\lambda\mu}=\Gamma^\sigma{}_{\lambda\alpha}g_{\sigma\mu}+\Gamma^\sigma{}_{\alpha\mu}g_{\lambda\sigma},$$

for given asymmetric $g_{\lambda\mu}$, should define $\Gamma^\nu{}_{\lambda\mu}$ uniquely. In the previous paper these conditions were given only in a very highly complicated form. The present paper is restricted to the case when $n=4$ and the symmetric part $h_{\lambda\mu}$ of $g_{\lambda\mu}$ has the relativistic signature $(---+)$, and the conditions are obtained in terms of the scalars $G=g/h$, $K=k/h$. Here g, h are the determinents $|g_{\lambda\mu}|$, $|h_{\lambda\mu}|$, and k is the determinant of the skew-symmetric part $k_{\lambda\mu}$ of $g_{\lambda\mu}$. Two essentially distinct cases arise according as the scalar density

$$D\equiv(\tfrac{1}{2}hk_{\lambda\mu}k^{\lambda\mu})^2-4hk$$

(suffixes being raised by means of $h^{\lambda\mu}$) is not, or is, identically zero. The main results are that the equations (*) admit a unique solution if, and only if, either (a) $D\neq 0$ and $G[\{2(1+3K)-G\}^2-4K\{3+K-G\}^2]\neq 0$; or (b) $D=0$.

H. S. Ruse (Leeds).

Hlavatý, Václav. The spinor connection in the unified Einstein theory of relativity. Proc. Nat. Acad. Sci. U. S. A. **39**, 501–506 (1953).

This is an outline, without proofs, of a spinor theory associated with a space X_4 of fundamental tensor $g_{\lambda\mu}=h_{\lambda\mu}+k_{\lambda\mu}$ ($k_{\lambda\mu}$ symmetric, $k_{\lambda\mu}$ skew-symmetric), $h_{\lambda\mu}$ being of signature $(---+)$. A vector v_λ of X_4 is called isotropic if $h_{\lambda\mu}v^\lambda v^\mu=0$. The spin-space associated with any given point P of X_4 is a projective 3-space S_3 defined in such a way that, to any anisotropic vector v^λ at P in X_4 there corresponds a biaxial involution in S_3, the axes of which coincide when v^λ is isotropic. For variable P in X_4, the mapping of the local spin-spaces upon one another gives rise to a connection $\Gamma^a{}_{b\lambda}$, the indices a, b belonging to the spin-space and λ to X_4. If this is assumed to operate in X_4 by means of the Christoffel symbols of $h_{\lambda\mu}$, then $\Gamma^a{}_{b\lambda}$ is uniquely defined.

H. S. Ruse (Leeds).

Hlavatý, Václav. Connections between Einstein's two unified theories of relativity. Proc. Nat. Acad. Sci. U. S. A. **39**, 507–510 (1953).

Of the theories referred to in the title, the first [Einstein, Ann. of Math. (2) **46**, 578–584 (1945); Rev. Modern Physics **20**, 35–39 (1948); these Rev. **7**, 266; **10**, 157; Einstein and Strauss, Ann. of Math. **47**, 731–741 (1946); these Rev. **8**,

412] has a complex fundamental tensor ${}^*g_{\lambda\mu}$ and a complex connection ${}^*\Gamma^\nu{}_{\lambda\mu}$, both hermitian-symmetric, such that

$$ {}^*g_{\lambda\mu} = h_{\lambda\mu} + ik_{\lambda\mu}, $$
$$ \partial_\omega {}^*g_{\lambda\mu} = {}^*\Gamma^\alpha{}_{\lambda\omega} {}^*g_{\alpha\mu} + {}^*\Gamma^\alpha{}_{\omega\mu} {}^*g_{\lambda\alpha}, $$

where $h_{\lambda\mu}$ is symmetric and $k_{\lambda\mu}$ skew-symmetric. The second theory [Einstein, The meaning of relativity, 3rd ed., Princeton, 1950; these Rev. 14, 97] is characterized by real $g_{\lambda\mu}$, $\Gamma^\nu{}_{\lambda\mu}$ such that

$$ g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, $$
$$ \partial_\omega g_{\lambda\mu} = \Gamma^\alpha{}_{\lambda\omega} g_{\alpha\mu} + \Gamma^\alpha{}_{\omega\mu} g_{\lambda\alpha}. $$

Identifying the $h_{\lambda\mu}$, $k_{\lambda\mu}$ of the second set of formulae with those of the first, the author obtains the relationship between ${}^*\Gamma^\nu{}_{\lambda\mu}$ and $\Gamma^\nu{}_{\lambda\mu}$. In the case when the solution is unique, it turns out that ${}^*\Gamma^\nu{}_{\lambda\mu} = \Gamma^\nu{}_{\lambda\mu} - (i+1)\Gamma^\nu{}_{[\lambda\mu]}$ + certain other terms depending upon $h_{\lambda\mu}$, $k_{\lambda\mu}$ and upon the covariant derivative of $k_{\lambda\mu}$ with respect to the Christoffel symbols of $h_{\lambda\mu}$.

H. S. Ruse (Leeds).

Pastori, Maria. Sull'ufficio del tensore fondamentale nell'ultima teoria di Einstein. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 509–518 (1951).

The author lists some formal results in the calculus arising in Einstein's generalized theory of relativity where a non-symmetric tensor is used for raising and lowering indices and a non-symmetric affine connection is available. The author gives the commutator of the operations: (1) raising (or lowering) an index and (2) covariant differentiation of various types. *A. H. Taub* (Urbana, Ill.).

Lamson, K. W. On the curvature tensor of Einstein's generalized theory of gravitation. Canadian J. Math. **5**, 297–300 (1953).

The author uses a result due to Schouten to decompose a curvature tensor derived from a non-symmetric affine connection into the sum of five tensors with various symmetry properties. He then shows that the field equations proposed by Einstein in the generalized theory of gravitation [The meaning of relativity, 3rd ed., Princeton, 1950; these Rev. 14, 97] imply certain algebraic conditions on these five tensors. *A. H. Taub* (Urbana, Ill.).

Lichnerowicz, A. Sur les équations relativistes de la gravitation. Bull. Soc. Math. France **80**, 237–251 (1952).

The author reviews recent work by himself, Y. Fourès-Bruhat [C. R. Acad. Sci. Paris, **230**, 618–620; **231**, 318–320 (1950); Acta Math. **88**, 141–225 (1952); these Rev. **11**, 548; **12**, 185; **14**, 756] and Y. Thiry [ibid. **226**, 1881–1882 (1948); J. Math. Pures Appl. (9) **30**, 275–316, 317–396 (1951); these Rev. **10**, 158; **13**, 787] on the solutions of the Einstein gravitational field equations. The method used by Fourès-Bruhat for obtaining an existence theorem for the solutions of these field equations subject to certain restrictions on the initial data is outlined. The author also summarizes the proof of his own theorems including the following one: If space-time admits a time-like one-parameter group and if the three spaces orthogonal to the trajectories of this group are compact and everywhere regular, then the space-time is locally Euclidean. *A. H. Taub* (Urbana, Ill.).

Bonnor, W. B. Certain exact solutions of the equations of general relativity with an electrostatic field. Proc. Phys. Soc. Sect. A. **66**, 145–152 (1953).

This paper considers the field equations of general relativity for empty space in which an electrostatic field is present. Several rigorous solutions of these equations are obtained some of which correspond to those previously obtained by Weyl. A physical interpretation of the solutions is given. *M. Wyman* (Edmonton, Alta.).

Buchdahl, H. A. On a set of conform-invariant equations of the gravitational field. Proc. Edinburgh Math. Soc. (2) **10**, 16–20 (1953).

Let g_{ij} be the components of the metric tensor in a four-dimensional space and G_{ij} the components of the Ricci tensor based on the above metric tensor. The author considers the invariant $K = 3G_{ij}G^{ij} - G_i{}^i G_j{}^j$ and proves the following theorem. "If p^{ij} denotes the Hamiltonian derivative of K then the equations $p^{ij} = 0$ possess as a solution every line element representing a space conformal to an Einstein space." In addition to proving this theorem this paper deals at some length with the particular case of static spherically symmetric fields. *M. Wyman.*

Nariai, Hidekazu. On a new cosmological solution of Einstein's field equations of gravitation. Sci. Rep. Tôhoku Univ. Ser. I. **35**, 62–67 (1951).

The author obtains several special spherically symmetric solutions of Einstein's field equations for empty space. It should be pointed out that the reviewer obtained the general solution for this case [Physical Rev. (2) **66**, 267–274 (1944); these Rev. **6**, 241]. The solutions obtained in the present paper are all included as special cases of the general solutions. In fact there exist other solutions in terms of elementary functions which are not mentioned by the author. After discussing several solutions the author gives a detailed account of the physical properties of one of his particular solutions. *M. Wyman* (Edmonton, Alta.).

Nariai, Hidekazu. On some static solutions of Einstein's gravitational field equations in a spherically symmetric case. Sci. Rep. Tôhoku Univ. Ser. I. **34**, 160–167 (1950).

Using the isotropic form of a spherically symmetric static line element the author finds several new solutions of the field equations of General Relativity. The procedure used is a mathematical one in that certain mathematical assumptions are made that simplify the equations to a point where a solution can be obtained. Physical interpretations are not given for the new solutions obtained. *M. Wyman.*

Kustaanheimo, Paul, and Qvist, Bertil. Some remarks on Sendai Astronomiaj Raportoj nro 23. Soc. Sci. Fenn. Comment. Phys.-Math. **16**, no. 12, 3 pp. (1952).

The present note is a comparison between the methods of the authors and those of Nariai in the paper reviewed above [=Sendai Astronomiaj Raportoj nro 23] in obtaining solutions of Einstein's field equations. *M. Wyman.*

***Popovici, Andrei. Sur les équations unitaires de la gravitation et de l'électromagnétisme.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 665–672. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

This paper states in outline form a unified field theory for the gravitational and electromagnetic fields. It involves a twenty-four-dimensional space whose metric tensor is a linear combination of the metric tensor of six four-dimensional spaces, each of which is taken to satisfy the field equations of general relativity. *A. H. Taub.*

***Lampariello, Giovanni. Progressi recenti di elettrodinamica relativistica.** Atti del Quarto Congresso dell' Unione Matematica Italiana, Taormina, 1951, vol. I, pp. 232–251. Casa Editrice Perrella, Roma, 1953.

McCrea, W. H. The Fitzgerald-Lorentz contraction—some paradoxes and their resolution. Sci. Proc. Roy. Dublin Soc. (N.S.) **26**, 27–36 (1952).

A detailed examination of the nature of the FitzGerald-Lorentz contraction, and of the relation the various "lengths" which can be ascribed to a moving rod. The author confirms the orthodox view that the FitzGerald-Lorentz contraction does not imply any intrinsic change in a body moving with constant velocity, and goes further to examine what happens if an attempt is made to "trap" the body between two fixed scale divisions.

H. P. Robertson (Pasadena, Calif.).

Whitrow, G. J. The Fitzgerald-Lorentz contraction phenomenon and theories of the relativity of Galilean frames. Sci. Proc. Roy. Dublin Soc. (N.S.) **26**, 37–44 (1952).

Reviews the search for a criterion to distinguish Galilean —or Minkowskian—frames for the expression of physical law. The author maintains that such a criterion is to be found in a modification, herein proposed, of Milne's kinematical relativity. *H. P. Robertson* (Pasadena, Calif.).

Stephenson, G., and Kilmister, C. W. A unified field theory of gravitation and electromagnetism. Nuovo Cimento (9) **10**, 230–235 (1953).

The authors derive the usual field equations for gravitational and electromagnetic fields from the metrical properties of a non-Riemannian space-time structure. The line element is made to depend on a symmetric tensor and a four vector. *M. Wyman* (Edmonton, Alta.).

Stephenson, G. Affine field structure of gravitation and electromagnetism. Nuovo Cimento (9) **10**, 354–355 (1953).

The authors investigate a possible extension of the work of the paper reviewed above in which the affine properties of the non-Riemannian space-time are considered. They conclude that the metrical approach of their previous paper is more satisfactory than their present extension.

M. Wyman (Edmonton, Alta.).

Moore, Marvin G. A quasi-relativistic theory of gravitation. Physical Rev. (2) **89**, 587–590 (1953).

In this work a theory of the interaction of gravitons and photons is given. The square of the velocity of light is supposed to be a harmonic function. Two cases are distinguished: (1) the plane of polarization passes through the sun, i.e., it involves the gravitational ray; (2) the plane of polarization is perpendicular to that of (1). For unpolarized rays we get from the above assumption (for m/r small) the Schwarzschild equation of the general theory of relativity.

L. Infeld (Warsaw).

Iwata, Giiti. Realization of special contact transformations with static electromagnetic fields in vacuo. II. Progress Theoret. Physics **9**, 97–107 (1953).

Continuing a previous paper [same journal **8**, 183–192 (1952); these Rev. **14**, 603] the author considers solutions $x_r(t, a_s, b_s)$, $y_r(t, a_s, b_s)$ of the Hamiltonian equations $dx_r/dt = \partial H/\partial y_r$, $dy_r/dt = -\partial H/\partial x_r$, where a_s, b_s are initial values of x_s, y_s. He notes that the contact transformation $(a_s, b_s) \to (x_s, y_s)$ is characterized by the eiconal matrix

$$S = \begin{Vmatrix} \partial x_r/\partial a_s & \partial x_r/\partial b_s \\ \partial y_r/\partial a_s & \partial y_r/\partial b_s \end{Vmatrix} = \begin{Vmatrix} s_{ra} & s_{rb} \\ s_{ya} & s_{yb} \end{Vmatrix}$$

and that, if s_{ra} or s_{rb} vanishes, we have a momentum-coordinate or coordinate-coordinate transformation, resp.; S satisfies $dS/dt = KS$, where

$$K = \begin{Vmatrix} \partial^2 H/\partial y_r \partial x_s & \partial^2 H/\partial y_r \partial y_s \\ -\partial^2 H/\partial x_r \partial x_s & -\partial^2 H/\partial x_r \partial y_s \end{Vmatrix}$$

and is symplectic in the sense that $S^* JS = J$, where * means transpose and

$$J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

He proceeds to discuss momentum-coordinate and coordinate-coordinate (or stigmatic) transformations more fully than in the earlier paper, and treats also the relativistic motion of a particle in a given electromagnetic field, using the variational principle $\delta \int \Omega_4(x) = 0$, where

$$\Omega_4(x) = -mc(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2)^{\frac{1}{2}} + \frac{e}{c}(A_0 dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3);$$

he finds that a relativistic stigmatic transformation is a Lorentz transformation, the change in the potential 4-vector A_r being a gauge transformation. [The transition from Newtonian mechanics to relativity makes an important difference in the problem of stigmatic transformations. In Newtonian mechanics we have, for a particle, ∞^3 trajectories through an initial position in space, and the condition for a stigmatic image is that all these trajectories should intersect at a single point at the same time t. In relativity, we have ∞^3 curves through an initial event in space-time, and the condition for a stigmatic image is simply that all these curves should intersect at a second event. The Newtonian case requires Hamilton's dynamical method, the parameter t playing an essential part, but the relativistic case lends itself to Hamilton's optical method, the extension from three to four dimensions being direct. Thus the relativistic stigmatic problem of the paper under review is essentially that of the absolute optical instrument [cf. C. Carathéodory, Geometrische Optik, Springer, Berlin, 1937, p. 70; and J. L. Synge, Trans. Amer. Math. Soc. **44**, 32–46 (1938)].] *J. L. Synge* (Dublin).

Pham, Man Quan. Thermodynamique d'un fluide relativiste. C. R. Acad. Sci. Paris **236**, 2299–2301 (1953).

In the space-time of general relativity with line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ (Greek suffixes = 0, 1, 2, 3, misprinted as 1, 2, 3 in the paper) the author takes for the energy tensor of a thermodynamic fluid

$$(1) \qquad T_{\alpha\beta} = \rho u_\alpha u_\beta - \pi_{\alpha\beta} - (u_\alpha q_\beta + u_\beta q_\alpha)$$

where ρ = density, $\pi_{\alpha\beta}$ = stress, u_α = unit 4-velocity and q_α = heat-flow vector, this last being connected with temperature (θ) by

$$(2) \qquad q_\alpha = -k \partial_\rho \theta (g_\alpha{}^\rho - u^\rho u_\alpha),$$

where k is a thermodynamic coefficient; hence $u^\alpha q_\alpha = 0$, so that the stream lines are orthogonal to the lines of heat flow. The condition $u^\alpha \pi_{\alpha\beta} = 0$ is also imposed, and an equation of heat conduction

$$(3) \qquad \nabla_\alpha q^\alpha = C \rho u^\alpha \partial_\alpha \theta,$$

where C is a specific heat; this last equation, the author claims, can be established by a detailed study (it appears to the reviewer that some extra hypothesis must underlie this, unless (3) is regarded as a definition of C). The equations of the fluid are furnished by the conservation equations $\nabla_\alpha T^{\alpha\beta}=0$, with (2) and (3). For a perfect thermodynamic fluid the author takes $\pi_{\alpha\beta}=p(g_{\alpha\beta}-u_\alpha u_\beta)$, where p = pressure, so that

$$(1') \qquad T_{\alpha\beta}=(\rho+p)u_\alpha u_\beta-pg_{\alpha\beta}-(u_\alpha q_\beta+u_\beta q_\alpha).$$

The conservation equations, with (2), (3) and $u^\alpha u_\alpha=1$, then provide 10 equations involving the 11 quantities ρ, p, θ, u_α, q_α, the 10 quantities $g_{\alpha\beta}$, and the two parameters k, C.

J. L. Synge (Dublin).

Pham, Man-Quan. Le problème de Cauchy pour un fluide parfait thermodynamique. C. R. Acad. Sci. Paris **237**, 22–24 (1953).

In this paper the author uses the equations (1'), (2), (3) of the paper reviewed above, together with an equation of state $\rho=\phi(p,\theta)$ and the field equations

$$S_{\alpha\beta}\equiv R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R=\chi T_{\alpha\beta}.$$

It is assumed that $g_{\alpha\beta}$, $\partial_\lambda g_{\alpha\beta}$, θ, $\partial_\lambda\theta$ are given (Cauchy data) on a hypersurface S, locally defined by $x^0=0$, and the aim of the paper is to show that these data suffice to determine the gravitational field and the behavior of the fluid in the neighborhood of S. The field equations are equivalent to the two systems ($i, j=1, 2, 3$)

$$R_{ij}=\chi[(\rho+p)u_iu_j-\tfrac{1}{2}(\rho-p)g_{ij}-(u_iq_j+u_jq_i)],$$
$$S_\lambda{}^0=\chi[(\rho+p)u^0u_\lambda-pg_\lambda{}^0-(u^0q_\lambda+q^0u_\lambda)],$$

where, on account of the Cauchy data for $g_{\alpha\beta}$, $S_\lambda{}^0$ are known on S; this splitting of the field equations is the key to the argument [cf. E. Cartan, Bull. Soc. Math. France **59**, 88–118 (1931); A. Lichnerowicz, Problèmes globaux en mécanique relativiste, Hermann, Paris, 1939; these Rev. **1**, 282; see also A. Lichnerowicz, Étude mathématique des théories relativistes de la gravitation et de l'électromagnetisme, t. I, Collège de France, Paris, 1953]. The result is established by manipulating the equations so that they can be solved for ρ, p, $\partial_0 p$, $\partial_{00}\theta$, u_α, $\partial_0 u_\alpha$, $\partial_{00}g_{\alpha\beta}$ on S in terms of the Cauchy data; hence all derivatives on S can be calculated in terms of the data, so that, but for exceptional cases, the Cauchy problem admits a solution, a result which the author regards as helping to justify the equations which he uses. In particular he remarks that the linkage (raccordement) of the field of a perfect thermodynamic fluid with a vacuum (external field) may be effected across a hypersurface S on which $\theta=$ const., $\partial_\alpha\theta=0$, with S composed of stream lines and $p=0$ on S. (The status of the parameters k and C (see preceding review) is not clear to the reviewer; are they assumed to be constants or given functions of p and θ? In either case, it may be noted, the system of equations has the correct 4-fold indeterminacy corresponding to arbitrary coordinate transformations; if k, C are constants, there are 17 equations in all (10 field equations, (2), (3), $u^\alpha u_\alpha=1$, and the equation of state) and 21 unknowns; if k, C are given functions of p, θ, there are two more equations and two more unknowns.)

J. L. Synge (Dublin).

Synge, J. L. Effects of acceleration in the Michelson-Morley experiment. Sci. Proc. Roy. Dublin Soc. (N.S.) **26**, 45–54 (1952).

An attempt to account, within the theory of relativity, for the fringe shifts reported by D. C. Miller, in his repetition of the Michelson-Morley experiment, as due to the effects of acceleration on a "physically rigid" interferometer. Starting from the fact that in relativity there can exist no mathematically rigid body, in the sense of Newton, the author develops the consequences for the problem at hand of a definition of a relativistically rigid body proposed by G. H. F. Gardner. Synge finds that fringe shifts, of about four times the magnitude of those given by Miller, would be expected if the interferometer arms were firmly coupled to the earth, a condition which he considers partially satisfied in the case of Miller's experiment. An alternative arrangement of the Michelson-Morley, designed to bring out the full magnitude of the predicted shift, is proposed.

H. P. Robertson (Pasadena, Calif.).

Graef Fernández, Carlos. Variational principles for the movement of the planets in a central field of Birkhoff. Revista Mexicana Fisica **2**, 180–192 (1953). (Spanish)

Let (t, x, y, z) be coordinates in a Minkowski space-time. Let $r=\sqrt{(x^2+y^2+z^2)}$ and $v=\sqrt{(\dot{x}^2+\dot{y}^2+\dot{z}^2)}$, a dot denoting the t-derivative. Then the equations of motion of a test-particle in the gravitational field of a central mass, are, according to Birkhoff's theory,

$$\ddot{x}=M(-x-xv^2+2\dot{x}r\dot{r})/r^3, \text{ etc.},$$

units being chosen so as to make the velocity of light c and the gravitational constant G both equal to unity. The author shows that these equations arise from the variational principle

$$\delta\int Ldt=0, \quad L=\tfrac{1}{2}me^{2M/r}(1+v^2),$$

m being the mass of the test-particle. To a first approximation this Lagrangian is equal to that of the corresponding Newtonian theory. The world-lines of such test-particles, in the 4-dimensional space-time, are likewise the extremals of a variational principle.

H. S. Ruse (Leeds).

MECHANICS

***Mercalov, N. I. Teoriya prostranstvennyh mehanizmov. [The theory of spatial mechanisms.]** Pp. 3–124, 195–199. Gosudarstv. Naučno-Tehn. Izdat. Mašinostroit. Lit., Moscow, 1951. 205 pp. 11.40 rubles.

The title of this book is misleading. In reality, the book is a posthumous collection of nine of Mercalov's papers of which the six first ones (124 pp.) deal with mechanisms, while three (76 pp.) do not. Moreover, the part dealing with mechanisms is partly concerned with plane kinematics or quite elementary theorems of spatial kinematics. Papers 7 and 8 (pp. 125–194) concern the hydrodynamic theory of lubrication and are reviewed separately [these Rev. **14**, 1139].

The first paper (12 pp.) deals with the determination of the position of a skew seven-bar linkage by means of descriptive geometry. The second is a three-page remark on the velocity distribution in a seven-bar chain. The third paper (56 pp.) is mostly expository and has been compiled from the author's 1920–21 lectures on space kinematics by one of his students. It contains some further remarks about seven-bar space linkages. Papers four and five (46 pp.) treat gears with skew axes, and are mostly expository. Paper six

(6 pp.) suggests how to use a chain belt to produce a variable transmission ratio between two parallel axles (when unround gears are inconvenient).

The ninth paper (5 pp.) considers the motion of a rigid body with a fixed point at which $A=B=4C$, the mass center being located in the equatorial plane of the inertial ellipsoid. He derives two equations for $\omega=(p^2+q^2)^{1/2}$ and q, containing the energy constant and the constant vertical momentum component, from the energy equation and the vertical momentum equations. One of these equations is integrable again if $A=4C$, and an equation for ω alone can be obtained. The approximate solution for q and ω by means of a two-term polynomial in $\zeta=\int p dt$ is given. [The dependence of ζ on t will have to be determined by another integration.]

A. W. Wundheiler (Chicago, Ill.).

Serebrennikov, M. G. Cam profiles for flat followers. Akad. Nauk SSSR. Inženernyĭ Sbornik **12**, 23–36 (1952). (Russian)

Let $L(u)=x\cos u+y\sin u-p(u)$ where u is the angle of the cam with the x-axis, and p the distance of a rectilinear flat follower from the center of the cam. Then, obviously, the cam profile C is the envelope of the line $L(u)=0$, its involute C' is the envelope of $L'(u)=0$, the involute C'' of C' is the envelope of $L''(u)=0$, etc. This observation is credited to Meissner [Schweizerische Bauzeitung **62**, 199–203, 221–224 (1913); **84**, 273–276 (1924)], rejustified at great length, and connected with cam-design procedures. The remaining five pages discuss four special cases of $p(u)$.

A. W. Wundheiler (Chicago, Ill.).

Braunbek, W. Der symmetrische Kreisel mit zeitlich periodischem Richtmoment. Z. Angew. Math. Mech. **33**, 174–188 (1953). (English, French and Russian summaries)

This paper is concerned with the motion of a symmetrical top subject to a torque expressible as the vector product of a unit vector in the direction of the axis of figure and a vector $A+B$, where A is a constant vector and B is a vector which varies periodically with respect to time. Special cases of the general problem are of interest in astronomy and atomic physics. The author gives extensive discussions of the cases in which B is, respectively, always collinear with A and always perpendicular to A. Many interesting phenomena are revealed, in particular a variety of resonance effects. The methods employed are simple, depending largely upon the selection of significant subcases, and the use of appropriate physically plausible approximations to simplify the differential equations of motion. Consequently, although the results appear to be entirely reliable, they include little that can be regarded as being established in a mathematically rigorous way. *L. A. MacColl* (New York, N. Y.).

Erdös, Paul. Kleine Schwingungen dynamischer Systeme. Z. Angew. Math. Physik **4**, 215–219 (1953).

Using concepts and methods of linear algebra, the author gives a new and compact formulation of the classical theory of small vibrations of a dynamical system about a configuration of equilibrium. *L. A. MacColl.*

Heinrich, G. Die Bedeutung des Schwingungsmittelpunktes für gewisse Verallgemeinerungen der Pendelbewegung. Österreich. Ing.-Arch. **7**, 117–124 (1953).

It is shown that, when the axis of a physical pendulum is forced to undergo an arbitrary translatory motion, the center of oscillation has the same motion as the mass of a similarly guided mathematical (simple) pendulum. The reactions upon the support are, in general, different. Similar relationships are derived to compare the motions and reactions of a physical conical pendulum with the mathematical conical pendulum. *M. Goldberg* (Washington, D. C.).

Tzénoff, Iv. Sur les théorèmes généraux du mouvement d'un corps solide par rapport à un système de coordonnées mobiles. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie II. **47**, 33–58 (1952). (Bulgarian. French summary)

After rederiving the equations of relative motion for a rigid body (in vector notation), the author applies them to the problem of the motion of a sphere on (A) a smooth or (B) a perfectly rough horizontal plane, with the apparent gravity assumed (a) constant or (b) composed of the earth's attraction and the centrifugal force. Four problems arise; they are related to those treated in articles 43–46 of vol. 2 of the much unquoted "Dynamics of a system of rigid bodies" [6th ed., Macmillan, London, 1930] by Routh. In the case (A) the equations are linear with constant coefficients, and the motion of the center is separated from that about the center. In the case (B) there is no separation, but the elimination of the reaction furnishes linear equations with constant coefficients for the center coordinates, and a closed solution can be obtained. *A. W. Wundheiler.*

Hydrodynamics, Aerodynamics, Acoustics

Castoldi, Luigi. Sulle derivate temporali fatte rispetto a diversi sistemi curvilinei e deformabili di riferimento dotati del più generale moto relativo. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 23–28 (1951).

Let x^i and y^i be two systems of (deformable) coordinates in a 3-space and t the time, so that to any event there correspond a tetrad (x^i, t) and a tetrad (y^i, t), so that a transformation $x^i=X^i(y, t)$ is assigned. The motion of a particle is defined by assigning y^i as functions of t. Then $f(x, t)$ being any assigned function, the derivatives df/dt and d^2f/dt^2 (following the motion of the particle) can be expressed in terms of the derivatives of the y's with respect to t, the partial derivatives of the X's with respect to the y's and t, and the partial derivatives of f with respect to the x's and t. The author works out these formulae and shows how they generalise the Coriolis formula for acceleration and also formulae familiar in the kinematics of continua.

J. L. Synge (Dublin).

Itô, Hirosi. Variational principle in hydrodynamics. Progress Theoret. Physics **9**, 117–131 (1953).

The author first reviews the classical formulation of barotropic flow of an inviscid fluid by means of a variational principle [Clebsch, J. Reine Angew. Math. **54**, 293–312 (1857); ibid. **56**, 1–10 (1859); Basset, A treatise on hydrodynamics, 2 vols., Deighton-Bell, Cambridge, 1888, see §§4, 34; H. Bateman, Proc. Roy. Soc. London. Ser. A. **125**, 598–618 (1929); Physical Rev. (2) **38**, 815–819 (1931)]. He next obtains a variational formulation for isentropic flow. $D\rho/Dt+\rho \operatorname{div} \mathbf{v}=0$ and $D(\rho S)/Dt+\rho S \operatorname{div} \mathbf{v}=0$, where S is the entropy, are in effect added as side conditions; the internal energy is used in the action integral, and the two multipliers ϕ and μ turn out to be Monge potentials for the velocity, $\mathbf{v}=-\operatorname{grad}\phi-\lambda \operatorname{grad}\mu$, where $\lambda=S$. In place of the

classical transformation of Clebsch ($D\lambda/Dt=0$, $D\mu/Dt=0$), there now results $D\lambda/Dt=0$, $D\mu/Dt=T$, where T is the temperature. [Cf. the variant of Clebsch's transformation in the barotropic case obtained by Ertel, S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Naturwiss. **1952**, no. 3; these Rev. **14**, 809.] The author obtains a similar formulation for isothermal flow. [These results may be compared with those of Taub, Proc. Symposia Appl. Math., v. 1, Amer. Math. Soc., New York, 1949, pp. 148–157 [these Rev. **11**, 222], who includes possible shock waves; with those of van den Dungen, Colloques Internat. Centre Nat. Recherche Sci. no. **14**, 88–95 (1949) [these Rev. **11**, 622], who uses material coordinates; with those of Dedecker, Inst. Roy. Météorolog. Belgique. Misc. no. **36** (1951) [these Rev. **13**, 1003], who obtains a 12-dimensional Hamiltonian form.]

The author uses his results to obtain a quantum hydrodynamics. He does not quantize the entropy, which he regards as "a geometrical parameter which plays a role to produce a vortex line in Clebsch's transformation". On the basis of his results he criticizes the theories of Landau [Acad. Sci. USSR. J. Phys. **5**, 71–90 (1941) and London Rev. Modern Physics **17**, 310–320 (1945)]. One of his results agrees with one of London's, but he leaves to the future "the somewhat mysterious relation between entropy and rotational motion". *C. Truesdell* (Bloomington, Ind.).

Masuda, Hideyuki. A new proof of Lagrange's theorem in hydrodynamics. J. Phys. Soc. Japan **8**, 390–393 (1953).

"Hydrodynamics is a branch of science which was developed at comparatively old times. The author wishes to review the method of derivation of its fundamental theorems, and to make use of the methods of solving linear partial differential equations to the possible maximum extent." The author considers circulation-preserving hydrodynamics as a problem in Hamilton-Jacobi theory. [The work is of interest in that it shows how awkward and indirect is the application of the general theory of partial differential equations to this problem, as compared with the several methods (all based on the existence of an intermediate integral of very simple form) which have developed in classical hydrodynamics.] *C. Truesdell* (Bloomington, Ind.).

Carstoiu, Ion. Sur la déformation d'une particule dans le mouvement d'un fluide. C. R. Acad. Sci. Paris **236**, 2209–2211 (1953).

Let d_{ij} be the rate-of-deformation tensor in a circulation-preserving motion; let $\chi=\int_{t_0}^{t}(Q+\frac{1}{2}v^2)dt$, where Q is the acceleration-potential and where the integration follows the path of particle; let x^i and X^α be spatial and material coordinates, let $x^i{}_{,\alpha}\equiv\partial x^i/\partial X^\alpha$, etc., and let $X_i{}^\alpha$ be the cofactor of $x^i{}_{,\alpha}$ in $\det x^i{}_{,\alpha}$; let $d^0{}_{\alpha\beta}$ and $V_\alpha{}^0$ be the values of d_{ij} and of the velocity for the particle in question when $t=t_0$. Then by means of Weber's transformation the author shows that

$$d_{ij}=\left(\frac{\rho}{\rho_0}\right)\left[d^0{}_{\alpha\beta}-\frac{\rho}{\rho_0}V_\gamma{}^0X_k{}^\gamma x^k{}_{,\alpha\beta}\right]X_i{}^\alpha X_j{}^\beta+\chi_{,ij}.$$

This elegant formula is the analogue of Cauchy's formula for the vorticity, and it is the general solution of Appell's differential equation for d_{ij} [J. Math. Pures Appl. (5) **9**, 5–19 (1903)]. *C. Truesdell* (Bloomington, Ind.).

Serrin, James. On plane and axially symmetric free boundary problems. J. Rational Mech. Anal. **2**, 563–575 (1953).

Ce sont de nouvelles applications des théorèmes de comparaison pour les sillages et les jets, plans symétriques, ou de révolution, qui ont fait l'objet de travaux récents de D. Gilbarg [même J. **1**, 309–320 (1952); ces Rev. **13**, 877] et J. Serrin [ibid. **1**, 1–48, 563–572 (1952); Amer. J. Math. **74**, 492–506 (1952); ces Rev. **13**, 877; **14**, 325]. Dans une première partie l'auteur établit que tout obstacle (canal) constitué par une courbe "en étoile" donne lieu à un sillage (jet) dont la ligne libre est nécessairement "en étoile", ce qui, en particulier, permet de généraliser l'énoncé d'un théorème de Gilbarg sur l'unicité des sillages à symétrie de révolution. Les sillages (jets) créés par des obstacles (canaux) différents, ayant certains éléments en commun, sont ensuite comparés, et cela conduit à une nouvelle démonstration de l'unicité des écoulements de ce type. Pour finir l'auteur détermine la forme de l'obstacle de dimensions données, qui rend minimum la résistance hydrodynamique. Il est montré que cet obstacle est obtenu à partir du schéma de Kirchhoff pour un segment rectiligne. *R. Gerber.*

Greene, Thom R., and Heins, Albert E. Water waves over a channel of infinite depth. Quart. Appl. Math. **11**, 201–214 (1953).

C'est l'extension au cas de la profondeur infinie des travaux récents de A. E. Heins [e.g., Canadian J. Math. **2**, 210–222 (1950); ces Rev. **11**, 696] sur la propagation des ondes de gravité dans un canal, où sont immergés différents types d'obstacles. Ici l'obstacle est un demi-plan horizontal, situé au-dessous de la surface libre, et la propagation est supposée ne pas être perpendiculaire à l'arrête du demi-plan. En linéarisant les équations du mouvement, on est conduit à la résolution d'un problème linéaire aux limites pour l'équation: $\phi_{xx}+\phi_{yy}=k^2\phi$. La solution de ce problème est donnée par l'intermédiaire d'une fonction de Green, et on en déduit, en particulier, la valeur des coefficients de réflection et de transmission. *R. Gerber* (Toulon).

***Birkhoff, Garrett, and Kotik, Jack. Fourier analysis of wave trains.** Gravity Waves, pp. 221–234. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

A statistical model for ocean swell is briefly described; this model is suggested by the principle of random phase (the proof of which will be published elsewhere): "Let Ω be any statistical ensemble of physical systems ω, each described by one or more function $f_\omega(t)$ of the time. If the statistics of Ω are time independent, then the phases $\epsilon_\omega(k)$ (k=frequency) are random and the phases of components having incommensurable frequencies are statistically independent". For each index m one selects an (almost periodic) function

$$f_m(t)=\frac{E(\infty)}{m}\sum_1^m\cos(k_i t-\epsilon_i),$$

where $E(k)$ is the energy spectrum of the swell, the frequencies k_i being selected at random with the probability distribution dE/dk and the phases ϵ_i by the principle of random phase.

By the central limit theorem the statistical distribution of the Fourier coefficients of $f_m(t)$ as $m\to\infty$ approaches a normal distribution, whose variance is computed. Applications of this method to the elevation of a seaway at a fixed point are given and compared to some experimental data.

J. Kampé de Fériet (Lille).

*Lacombe, H. **The diffraction of a swell. A practical approximate solution and its justification.** Gravity Waves, pp. 129–140. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

Wendt, H. Das Problem der Jungfernquelle. Z. Angew. Math. Mech. **32**, 338–358 (1952). (English, French and Russian summaries)

A source of strength e is placed at distance $h>0$ beneath the (undisturbed) surface of an infinite ocean. The motion is steady and axially symmetric, the water assumed inviscid and at rest at infinity. The author wishes to find the form of the free surface. Let $\psi(r, z)$ be the Stokes stream function, $u=-\psi_z/r$, $w=\psi_r/r$, z-axis upwards, z_M the ordinate of the free surface ($\psi(r, z_M)=0$). Then $\psi_{rr}-r^{-1}\psi_r+\psi_{zz}=0$ except at the source, the free surface condition is

$$[u(r, z_M)]^2+[w(r, z_M)]^2=-2gz_M,$$

and the necessary conditions at infinity must be satisfied. The author applies a method of successive approximations to treat this problem. The method involves expansion in powers of e and is presumably valid only for small e although convergence questions are not treated. Graphs of the first three approximations to z_M are shown for three choices of e and $h=1$. *J. V. Wehausen* (Providence, R. I.).

Apté, Achuyt. Sur une solution approchée du problème de l'onde solitaire. C. R. Acad. Sci. Paris **236**, 2477–2479 (1953).

The problem considered is that of the symmetrical "peaked" solitary wave in a channel. It is well known [cf. Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, p. 418] that the angle at the crest must be 120°. By an ingenious use of certain explicitly known conformal mappings the author is able to map a rectangular horizontal half-strip in the plane of the complex velocity potential, $f=\varphi+i\psi$, onto a region in the geometric plane bounded by the positive x-axis, a segment of the y-axis and a curve L having a horizontal asymptote and such that L makes an angle of 60° with the y-axis. This region and the mapping may be extended by symmetry, so that one has the desired 120° at the crest and a profile of the expected shape. The mapping depends on two parameters h and k, and the method consists in choosing values of h and k so that the boundary condition given by the Bernoulli law shall be satisfied as closely as possible along the free boundary L, or more precisely so that the quantity

$$\epsilon=|\tfrac{1}{2}v^2+gy-\tfrac{1}{2}v_\infty^2-gy_\infty|,$$

where $v=|df/dz|$, shall remain as small as possible along L.

By means of calculations which will be reported elsewhere the author finds values of h and k such that $\epsilon\leqq 0.3\%$ along L. With these values he finds the ratio of the height of the crest to that of the horizontal asymptote to be 1.844, compared with the result of 1.78 obtained previously by McCowan [Philos. Mag. (5) **38**, 351–358 (1894)].

D. H. Hyers (Los Angeles, Calif.).

*Kuessner, H. G. **A review of the two-dimensional problem of unsteady lifting surface theory during the last thirty years.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1953. i+22 pp. (mimeographed)

Truckenbrodt, E. Das Geschwindigkeitspotential der tragenden Fläche bei inkompressibler Strömung. Z. Angew. Math. Mech. **33**, 165–173 (1953). (English, French and Russian summaries)

It is shown how known formulas of lifting-surface theory can be obtained simply by considering the wing to be made up of elemental horse-shoe vortices. Induced velocity components and the velocity potential are expressed in terms of integrals of the bound circulation distribution, and several limiting cases are considered, viz., (a) points on the wing, (b) points far forward and far behind the wing, (c) the wing of infinite span (plane flow), (d) the wing of vanishing aspect ratio (R. T. Jones' "slender wing") and (e) the lifting line (Prandtl wing). *W. R. Sears* (Ithaca, N. Y.).

Pereira Gomes, A. Discussion d'un résultat de Tsien pour la détermination d'un "convergent". Choix de la distribution de vitesses sur l'axe. Gaz. Mat., Lisboa **14**, no. 54, 1–4 (1953).

The method of Tsien [J. Aeronaut. Sci. **10**, 68–70 (1943); these Rev. **5**, 80] for constructing contraction cones is discussed in certain cases. Tsien's solution contains an arbitrary function f which is the velocity distribution along the axis of the contraction. The present paper considers the case $f=a_0+a_1f_1$, where f_1 satisfies the differential equation $f_1''=\alpha^2f_1$ and a_0, a_1, α are constants. A generalization with a_1f_1 replaced by a finite sum of terms a_if_i, each with a different value of α, is also discussed. *P. W. Ketchum.*

Britten, K. H. V. The use of influence factors in problems of fluid flow. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2441**, 13 pp. (1952).

"A rigorous mathematical analysis has been applied to some empirical results, obtained by Thom, on the use of the 'squares method' in problems of fluid flow. The problem considered is that of determining the incompressible and compressible flow past an arbitrarily shaped body placed in a wind tunnel. The value of $\lim_{\phi'\to\infty}\{\phi'-x(\phi', \psi')\}/h$ (Thom's formula for incompressible flow) has been verified.... Thom's formulae for the influence factor F, which can be used to calculate the difference between the incompressible and compressible values of the potential function at infinity, have also been derived.

"The value of $y(\phi', \psi')$ has been obtained for incompressible flow round an arbitrary body in a wind tunnel of width $2h$, and it is shown that Thom's empirical formula 3.6.1 is correct only when $h\to\infty$." (The author introduces the velocity potential function, $\phi(x, y)$, and the stream function, $\psi(x, y)$, as independent variables and studies $(x+iy)$ as an analytic function of $(\phi+i\psi)$, for the incompressible case.) This paper is related to the one of A. Thom [Thom and Klanfer, same Rep. and Memoranda no. **2440** (1953), part I; these Rev. **14**, 922]. *E. Isaacson.*

Stewartson, K. On the slow motion of an ellipsoid in a rotating fluid. I. A formal solution. Quart. J. Mech. Appl. Math. **6**, 141–162 (1953).

The author considers the motion after an impulsive start of an ellipsoid in a rotating fluid with one of its principal axes parallel to the axis of rotation. It is assumed that the ellipsoid is initially at rest relative to the axes rotating with the fluid and that at a given instant it is given, impulsively, a velocity of translation and that thereafter it moves with given velocity and without relative rotation. By small

perturbation, the general solution is given in Part I. In Part II, the general solution is applied to the case of uniform motion, relative to the rotating axes, of an ellipsoid along and perpendicular to the axis of rotation. When the translation is parallel to the axis of rotation, it is found that in the flow field there is a cylindrical region in which the motion tends to a uniform translation and the streamlines are elliptic helices and never cross this cylinder. Outside this cylinder, there is a shearing motion parallel to the axis of rotation singular on the cylindrical surface and tending to zero at infinity. The whole motion is two-dimensional.

When the ellipsoid moves uniformly at right angles to the axis of rotation, the cylindrical region again exists. Outside the cylinder the motion is two-dimensional and rotational and is dependent on the semi-axis in the direction of rotation. The velocity is singular at the surface of the cylinder. Inside the cylinder the velocity is also singular and the streamlines relative to the ellipsoid lie in parallel planes through the axis of rotation. Thus, the streamlines which cross the cylinder make sharp turns at the surface of the cylinder. All these theoretical conclusions confirm Taylor's earlier observations [Proc. Roy. Soc. London. Ser. A. **93**, 99–113 (1917); **102**, 180–189 (1922); **104**, 213–218 (1923)]. The singularity on the cylinder is attributed to the effect of viscosity and an estimate based on boundary-layer theory is finally made. *Y. H. Kuo* (Ithaca, N. Y.).

Squire, H. B. Some viscous fluid flow problems. I. Jet emerging from a hole in a plane wall. Philos. Mag. (7) **43**, 942–945 (1952).

An exact axially symmetric solution of the Navier-Stokes equations is exhibited and interpreted as the jet flow of a viscous fluid emerging from an orifice in a slippery wall.

G. F. Carrier (Cambridge, Mass.).

Dean, W. R. Slow motion of viscous liquid near a half-pitot tube. Proc. Cambridge Philos. Soc. **48**, 149–167 (1952).

The Stokes linearization of the Navier-Stokes equations is invoked to investigate the problem wherein a viscous fluid in a channel of semi-infinite width impinges on a half-plane lamina oriented parallel to the flow direction. An approximate solution is found, and the wall pressure distribution prediction is compared to previously reported experiments. The discrepancy is of the order of 30%. *G. F. Carrier.*

***Mercalov, N. I. Teoriya prostranstvennyh mehanizmov. [The theory of spatial mechanisms.]** Pp. 125–194. Gosudarstv. Naučno-Tehn. Izdat. Mašinostroit. Lit., Moscow, 1951. 205 pp. 11.40 rubles.

This review treats only Chapters 7 and 8 on lubrication theory. The chapters on the theory of mechanisms are reviewed separately [these Rev. **14**, 1135].

In Chapter 7 the author treats the problem of the pressure distribution in a journal bearing due to the flow of lubricant between a pair of eccentric circles by solving the Navier-Stokes equations for the two-dimensional flow of a viscous fluid. This solution he attributes to Joukovsky and Chaplygin [cf. G. H. Wannier, Quart. Appl. Math. **8**, 1–32, (1950); these Rev. **12**, 217]. No references are given. Following this, the solution for the infinite journal bearing is worked out on the basis of Reynolds approximation.

In the next chapter the three-dimensional problem of the journal bearing is treated by a method also attributed to Chaplygin, though again no specific references are noted. A solution for the pressure distribution is obtained but this does not agree very well with experiment. This may be due to the approximations made or more likely to the difficulties entailed in carrying out the experiments to conform with the assumptions made at the start of the theoretical treatment. *E. Saibel* (Pittsburgh, Pa.).

Mihallov, G. K. On filtration in trapezoidal dikes with a vertical upstream slope. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 189–199 (1953). (Russian)

This is a continuation of the investigations of a filtration problem which were begun by P. Ya. Polubarinova-Kochina [Izvestiya Akad. Nauk SSSR. Ser. Mat. **1939**, 579–602; these Rev. **2**, 26] and which were continued by the author [Doklady Akad. Nauk. SSSR (N.S.) **80**, 553–556 (1951)]. Certain integrals, in terms of which the flow through the dam was expressed in the author's earlier paper, are here evaluated in terms of infinite series. It is shown that for special values of one of the parameters occurring in the problem, the given infinite series reduce to well-known formulas of hydraulics. *H. P. Thielman* (Ames, Iowa).

Pilatovskiĭ, V. P. On computation of the pressure function and discharge function in the case of filtration of an elastic fluid in a stratum. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 179–188 (1953). (Russian)

In this paper there is considered a method for the approximate solution of the problem of radial filtration of an elastic fluid through a layer which consists of concentric ring-shaped regions. In each of these regions the flow is assumed to be uniform. The method is based on the assumption that the solution can be represented as the sum of n terms of an asymptotic expansion (which is valid for a small initial interval of time) and of the remainder of the expansion, which is evaluated by means of a special trigonometric series. The method is applied to the solution of the problem of radial flow in a cylinder of finite radius on the boundary of which the discharge of the fluid is given.

H. P. Thielman (Ames, Iowa).

Nigam, Swami Dayal, and Rangasami, Kumandur Srinivasa Iyengar. Growth of boundary layer on a rotating sphere. Z. Angew. Math. Physik **4**, 221–223 (1953).

As a supplement to a recent paper of Howarth [Philos. Mag. (7) **42**, 239–243 (1951); these Rev. **12**, 871] the authors consider the boundary-layer equations for flow around a sphere rotating in fluid at rest. A solution for small t is found, where t denotes the time and the sphere is supposed to begin steady rotation suddenly at $t=0$. This solution exhibits radial inflow at high latitudes and outflow near the equator. *W. R. Sears* (Ithaca, N. Y.).

Gadd, G. E. Some aspects of laminar boundary layer separation in compressible flow with no heat transfer to the wall. Aeronaut. Quart. **4**, 123–150 (1953).

By the transformations of Crocco and Stewartson, the boundary-layer equations are first simplified. The enthalpy distribution is then calculated for the case of viscosity coefficient varying linearly with the absolute temperature T and the Prandtl number σ near unity by an approximate method similar to that of Howarth [Proc. Roy. Soc. London. Ser. A. **164**, 547–579 (1938)]. From the numerical analysis it was found that the problem of finding the separation point can be reduced to solving a total differential equation in one variable x measured along the surface of the body for σ

near unity. The case where $\sigma=1$ and the viscosity is proportional to T^{ω} and ω is near unity is similarly considered. The final conclusions are that, for Mach number as large as 10, the change in separation distance as compared with the case $\sigma=\omega=1$ is probably less than 40 per cent when $\omega=1$ and $0.7<\sigma<1$; and less than 25 per cent when $\sigma=1$ and $0.7<\omega<1$. *Y. H. Kuo* (Ithaca, N. Y.).

Dienemann, W. Berechnung des Wärmeüberganges an laminar umströmten Körpern mit konstanter und ortsveränderlicher Wandtemperatur. Z. Angew. Math. Mech. **33**, 89–109 (1953). (English, French, and Russian summaries)

Based on the improved Kármán-Pohlhausen method of Holstein and Bohlen [Lilienthal-Bericht **S10** (1941)], the author formulates the thermal problems in an analogous fashion. By defining a parameter $\Theta=\vartheta_t^2/\nu$ where ϑ_t and ν are respectively the heat-loss thickness and the kinematic viscosity coefficient, it is shown that the energy equation can be brought into a form similar to the momentum equation; viz., the streamwise variation of Θ can be expressed as a function of the Pohlhausen parameter, ratio between heat and momentum boundary-layer thicknesses, Prandtl number and the main stream velocity. In the case of arbitrary wall temperature distribution, a similar equation is also obtained. As numerical examples are treated both constant wall temperature and wall temperature varying according to $\cos(\pi x/l)$, x being measured from the leading edge and l the chord of the plate, at Prandtl numbers 0.7 and 10.

In the case of a yawed wing, following J. M. Wild's idea of heat analogy [J. Aeronaut. Sci. **16**, 41–45 (1949)], the problem of calculating the spanwise flow is formulated anew in accordance with the author's method outlined above. *Y. H. Kuo* (Ithaca, N. Y.).

Abdurahiman, P. V. Two-dimensional compressible flow past a solid body symmetrically placed in a channel. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2443** (**11,243**), 6 pp. (1952).

By Janzen-Rayleigh expansion, Goldstein and Lighthill have developed a method for studying the first-order compressibility effect on potential flow over a body placed either in an unlimited fluid or symmetrically in a parallel channel [Philos. Mag. (7) **35**, 549–568 (1944)]. In the present report, this method has been applied to the case of an oval placed symmetrically in a parallel channel. The maximum surface velocity has been calculated for different channel widths and thickness ratios. For each case a "blockage factor", i.e., the percentage increase of the maximum velocity due to the presence of walls, is calculated and numerically compared with the results of Thom [part I of Thom and Klanfer, Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2440** (1953); these Rev. **14**, 922]. *Y. H. Kuo* (Ithaca, N. Y.).

***Ludford, G. S. S. The extension to a two-dimensional subsonic flow of the Joukowski force of ideal fluid theory.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 715–721. The American Society of Mechanical Engineers, New York, N. Y., 1952.

L'auteur calcule les éléments de réduction du système des forces de pression exercées sur un obstacle par un écoulement plan, permanent, et subsonique d'un fluide compressible, de vitesse uniforme à l'infini, et dans lequel il n'y a pas de tourbillons. Les résultats permettent en particulier d'étendre à ces mouvements le théorème de Kutta-Joukowski, établi pour les fluides incompressibles. Le calcul repose essentiellement sur la connaissance du développement du potentiel des vitesses autour du point à l'infini. Dans un travail récent [J. Math. Physics **30**, 117–130 (1951); ces Rev. **13**, 399], l'auteur avait établi d'une manière rigoureuse ce développement, qui est de la forme

$$\frac{\phi}{U}=r\cos\theta+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f_{mn}(\theta)\left(\frac{\log r}{r}\right)^{m}\frac{1}{r^{n}}$$

où ϕ désigne le potentiel des vitesses, U la vitesse à l'infini, r et θ les coordonnées polaires. Ici ce résultat est justifié au moyen d'une méthode différente qui consiste à développer ϕ en une série de fonctions, définies par récurrence comme solution d'équations de Laplace à second membre.

R. Gerber (Toulon).

***Guderley, Gottfried. On the presence of shocks in mixed subsonic-supersonic flow patterns.** Advances in Applied Mechanics, vol. 3, pp. 145–184. Academic Press Inc., New York, N. Y., 1953. $9.00.

Ce mémoire contient l'exposé des recherches de l'auteur et de A. Busemann sur la question contreversée: existe-t-il en général des écoulements transsoniques continus? Les principaux arguments, tant mathématiques que physiques, avancés par ces auteurs pour justifier une réponse négative à cette question étaient déjà connus des spécialistes qui avaient pu avoir accès à certains rapports non officiellement publiés. Mais il est heureux de voir rassemblés ici de façon claire et complète les principaux développements de l'argumentation. L'étude est divisée en deux parties, l'une mathématique, l'autre physique.

Dans la partie mathématique, l'auteur, par une analyse simple des variations de pression le long d'une caractéristique au voisinage de la ligne sonique, montre d'abord qu'une région supersonique limitée dans l'écoulement autour d'un profil ne peut s'étendre en aval du profil, et que le contour limitant ce dernier ne peut contenir un segment de droite, si petit soit-il. Mais ceci n'est qu'une réponse partielle; une analyse plus fine est nécessaire, elle repose sur l'étude des problèmes aux limites relatifs à certaines équations aux dérivées partielles du type mixte, et en particulier sur l'impossibilité de définir une solution d'une telle équation dans une région mixte lorsqu'on impose des données tout le long du contour fermé limitant cette région. Comme l'étude mathématique de ces problèmes est actuellement peu développée, l'auteur doit utiliser les analogies et les cas particuliers. Si on part d'un écoulement à potentiel, transsonique, puis que l'on déforme légèrement le contour du profil dans le plan physique, on peut linéariser le nouveau problème par rapport au problème initial; dans le plan de l'hodographe on est ramené au problème suivant: trouver une solution d'une équation linéaire qui le long d'un contour fermé satisfait à une condition du type dérivée oblique assez simple; il faut alors montrer que ce problème n'admet pas de solution si la région ainsi limitée contient une partie hyperbolique. Pour simplifier l'auteur fait ces calculs en supposant valide l'équation de Tricomi, c'est-à-dire la validité des lois de similitude transsonique dans le cas étudié, et il montre la non-existence de la solution en étudiant des cas particuliers. Ceux-ci sont obtenus en considérant le problème dans des triangles mixtilignes particuliers, problème pour lequel les fonctions propres sont des solutions simples obtenues par séparation des variables (solutions

particulières de Darboux). Alors que le problème admet une solution finie et une seule dans le cas où le contour est elliptique, il n'admet aucune solution finie si le domaine limité est mixte. D'ailleurs la suite des valeurs propres présente des caractéristiques très différentes suivant que le domaine est totalement elliptique, totalement hyperbolique ou mixte. Certes, comme le reconnait l'auteur, ces raisonnements ne sont pas d'une rigueur mathématique absolument inattaquable car a) l'approximation de Tricomi est utilisée, b) l'impossibilité du problème mathématique envisagé n'est prouvée que pour des contours extrêmement particuliers; toutefois il est souvent légitime pour prouver la non-existence d'une solution de donner un contre-exemple.

La deuxième partie contient des considérations plus intuitives qui conduisent à des conclusions analogues. On envisage tout d'abord le développement dans la région supersonique d'une perturbation qui prend naissance le long d'un profil, par exemple une discontinuité de courbure; cette perturbation se propage vers l'aval le long d'une caractéristique qui se réfléchit successivement sur la ligne sonique et sur le profil, les arcs de caractéristique portant cette perturbation s'accumulant finalement au point de rencontre du profil et de la ligne sonique. On trouve également une analyse dans le plan physique de l'écoulement d'un tourbillon lorsqu'on impose une petite déformation à une ligne de courant; on est ainsi conduit à un problème de valeurs propres, et la suite de ces valeurs présente un caractère différent suivant la nature de l'écoulement. Enfin diverses interprétations physiques de la solution qui a servi à l'argumentation mathématique sont proposées et discutées.

P. Germain (Providence, R. I.).

Fenain, M. Recherche d'approximations des équations régissant les écoulements des gaz. Applications aux tuyères planes. Recherche Aéronautique no. 33, 11–28 (1953).

This paper belongs to a series by Germain and his associates (the bibliography gives other references) on the use of differential equations which, for the transonic flow of an ideal gas, approximate to the correct equation. The paper was communicated to the VIIIth International Congress of Theoretical and Applied Mechanics at Istanbul, 1952. The units of velocity V, speed of sound, density ρ and pressure being chosen to correspond to Mach number $M=1$, the hodograph variables V and θ (inclination of flow) are replaced by s, θ, where s is defined by $[f(s)]^{\frac{1}{2}}ds=-(1-M^2)^{\frac{1}{2}}dV/V$. When one writes $f(s)h^2(s)=(1-M^2)/\rho^2$, the differential equation for the stream-function ψ in the s, θ plane is $f(s)\psi_{ss}+\psi_{\theta\theta}+(h'/h)\psi_s=0$. This form gives more scope for approximations than the usual (Chaplygin) form which corresponds to $s=\sigma$, $h(s)\equiv 1$.

The author first seeks forms of equation having solutions which are constant along the tangents to the characteristics in the s, θ plane. He next illustrates a method of deriving solutions of new approximating differential equations of mixed type when suitable solutions of approximations to the simpler (σ,θ) equation are known. He shows how to transform the (s,θ) equation to obtain similar forms with more parameters; these parameters are available for improving the approximation to the equation of state which is implied by an approximating differential equation. The paper ends with an application to flow in a nozzle from subsonic to supersonic speed.

D. C. Pack (Glasgow).

Ludloff, H. F., and Friedman, M. B. Diffraction of blasts by axisymmetric bodies. J. Aeronaut. Sci. 19, 425–426 (1952).

The interaction of a plane shock and a slender axisymmetric body of arbitrary profile is discussed. The solution to the linearized problem of the pressure distribution is found in the form of a retarded potential integral. The special case where the body is a cone is specifically discussed.

G. F. Carrier (Cambridge, Mass.).

Murthy, S. N. B. On the characteristic method for unsteady compressible flow. Proc. Indian Acad. Sci. Sect. A. 37, 664–680 (1953).

The author considers the use of the method of characteristics for the graphical solution of one-dimensional unsteady flows relating, in particular, to internal combustion engines. The chief example concerns the flow out of a Diesel engine cylinder through a valve and nozzle. *D. Pack.*

Flax, A. H. Integral relations in the linearized theory of wing-body interference. J. Aeronaut. Sci. 20, 483–490 (1953).

The reverse flow theorems for a plane wing [F. Ursell and G. H. Ward, Quart. J. Mech. Appl. Math. 3, 326–348 (1950); these Rev. 14, 218; A. H. Flax, J. Aeronaut. Sci. 19, 361–374 (1952); these Rev. 14, 218] are extended to a plane wing mounted on a cylindrical body of arbitrary cross-section. Also, theorems expressing the total lift and rolling moment on the body in terms of the corresponding wing quantities are developed by a Trefftz plane analysis. These theorems then are used to establish several reciprocity relations such as: "the lift induced on the wings by a unit body angle of attack is equal to the lift induced on the body by a unit wing angle of attack in reverse flow."

J. W. Miles (Los Angeles, Calif.).

***Bartels, R. C. F., and Laporte, O. An application of conformal mapping to problems in conical supersonic flows.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 91–103. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

An approach to the theory of supersonic conical fields is described, in which the problem is reduced to the determination of a single function of a complex variable by conformal mapping methods. In other respects it differs from the method of Goldstein and Ward [Aeronaut. Quart. 2, 39–84 (1950)] which was considerably simpler as a result of working in a half-plane rather than an annulus, and working with the derivative of the basic function so as to avoid the appearance of elliptic functions or integrals until the very end of the investigation. *M. J. Lighthill* (Manchester).

Dunn, D. W., and Lin, C. C. On the role of three-dimensional disturbances in the stability of supersonic boundary layers. J. Aeronaut. Sci. 20, 577–578 (1953).

Müller, Wilhelm. Die Energiegleichungen für die Wärmeübertragung bei der kompressiblen Flüssigkeits- und Gasströmung mit Reibung. Österreich. Ing.-Arch. 7, 77–87 (1953).

The author's objective is to obtain the energy equation for fluids with the "utmost completeness possible". His general results are included among those of Vazsonyi [Quart. Appl. Math. 3, 29–37 (1945); these Rev. 7, 226] and the reviewer [Physical Rev. (2) 73, 513–515 (1948);

these Rev. 9, 474]. The paper ends with a derivation of the linearized absorption coefficient for plane acoustic waves under the unnecessary assumption that the fluid is a perfect gas [cf. P. Biquard, Ann. Physique (11) 6, 195–304 (1936)].

C. Truesdell (Bloomington, Ind.).

Brehovskih, L. M. On a case of propagation of sound in a nonhomogeneous medium. Doklady Akad. Nauk SSSR (N.S.) 87, 715–718 (1952). (Russian)

A point sound source is situated in the half-space $z>0$, where the plane $z=0$ is perfectly reflecting, the speed of sound being constant ($=c_0$) for $0\leqq z\leqq h$, and for $z\geqq h$ being $c=c_0(1+2a(z-h))^{-\frac{1}{2}}$. The author compares calculations of attenuation at large distances based on his general theory [Izvestiya Akad. Nauk SSSR. Ser. Fiz. 13, 505–514, 515–533, 534–545 (1949); these Rev. 11, 563, 564], on the work of Pekeris [J. Acoust. Soc. Amer. 18, 295–315 (1946); these Rev. 8, 239; 9, 735] and R. W. Morse [ibid. 22, 857–860 (1950)] for slightly different layer laws. He finds agreement with Pekeris for the case $h=0$, but disagrees numerically with Morse on the effect of a thick layer.

F. V. Atkinson (Ibadan).

Eckart, Carl. The scattering of sound from the sea surface. J. Acoust. Soc. Amer. 25, 566–570 (1953).

It has been noted that the study of scattered sound or other radiation by the sea surface can furnish information concerning the geometry and kinematics of that surface. In order that such information may be obtained from the experimental data, a theoretical analysis of the problem is carried out here. The analysis is difficult, and progress is made only by making plausible approximations.

The analysis is based on Helmholtz's formula which gives a solution of the equation of monochromatic waves as a surface integral. When this formula is applied to give the secondary (reflected plus scattered) pressure in the sound waves, assumptions have to be made as to the relation at the sea surface between the primary and secondary waves. It is assumed that the equation of the sea surface is $z=\zeta(x, y, t)$ and that the time average

$$\Phi=\langle\zeta(x', y', t)\zeta(x'', y'', t)\rangle$$

is a function of $\xi=x''-x'$ and $\eta=y''-y'$ alone. The scattering coefficient of long-wave sound is shown to be

$$\sigma=(c^2k^2/4\pi)^2\int\int\Phi(\xi, \eta)\exp\left[-ik(a\xi+b\eta)\right]d\xi d\eta,$$

where $2\pi/k$ is the wave length and a, b, c are respectively the sums of the x, y, z direction cosines of the incident and scattered rays. For short-wave sound, the formula, which does not involve the wave length, is much more complicated.

It turns out that experiments with short waves yield much less information about the sea surface than do those with longer waves, an unfortunate result since long wave experiments are much more difficult to perform.

E. T. Copson (St. Andrews).

Somers, Edward V. Propagation of acoustic waves in a liquid-filled cylindrical hole, surrounded by an elastic solid. J. Appl. Phys. 24, 515–521 (1953).

The author studies the radiation of acoustic waves into a liquid-filled circular tube of infinite extent inside an infinite elastic solid, by a time-harmonic, impulsive, radially symmetric, ring-shaped source lying in the tube and coaxial with it. The various wave potentials are expressed in terms of Fourier integrals (with respect to the axial coordinate), the integrands depending on Bessel functions I and K. The integrals are then transformed into sums of residues and contributions of branch cuts (discrete and continuous spectra respectively). The latter are evaluated asymptotically for large distances from the source. Extension to an arbitrary time-dependence of the source is included.

C. J. Bouwkamp (Eindhoven).

Voĭt, S. S. Propagation of waves from a radiating disc in a moving medium. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 699–705 (1952). (Russian)

A piston oscillates parallel to the z-axis through a hole in the infinite plane wall $z=0$, the infinite medium in $z>0$ flowing parallel to the x-axis. The acoustic potential at any point of the medium is given as a double integral, effectively the Lorentz transform of that investigated by D. N. Četaev [Doklady Akad. Nauk SSSR (N.S.) 76, 813–816 (1951); these Rev. 12, 650], which is approximated by the stationary phase method in two dimensions, for simplicity for points on the plane of symmetry. There is a principal wave in a certain cylinder; secondary terms are also discussed.

F. V. Atkinson (Ibadan).

Voĭt, S. S. Reflection and refraction of spherical sound waves in passing from a fixed to a moving medium. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 157–164 (1953). (Russian)

A harmonic point sound source is situated at $(0, 0, h)$ where the half-space $z>0$ is filled with a fixed uniform medium, and the distinct medium in $z<0$ moves uniformly parallel to the x-axis. The boundary problem is solved by the Weyl plane-wave method; boundary conditions at infinity are not mentioned. The double integrals for the reflected and refracted fields are approximated by the stationary phase method in two dimensions, substantially complicated here by singularities in the integrands. The leading terms, valid for large distances, resemble waves from image sources.

F. V. Atkinson (Ibadan).

Kay, Irvin. Diffraction of an arbitrary pulse by a wedge. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. EM-43. i+24 pp. (1952).

The author investigates characteristic boundary-value problems occurring in the problem of the title. He points out that the method of conical flow [J. B. Keller and A. Blank, Comm. Pure Appl. Math. 4, 75–94 (1951); same Mathematics Research Group, Rep. No. EM-21 (1950); these Rev. 13, 304; 12, 564] cannot in general be used for incident pulses of arbitrary shape. The author's solution of the general problem is obtained in the form of a Fourier series whose coefficients are determined by the application of an interesting integral transformation to the given boundary data on the characteristic cone. The major part of the paper is concerned with the derivation of this, possibly new, integral transform containing a Whittaker function and a modified Bessel function. It is verified that the plane-wave case of Keller and Blank [loc. cit.] is included in the general solution. There is an appendix by J. B. Keller on the diffraction of a shock or an electromagnetic plane pulse by a right-angled wedge, giving numerical results based on Keller and Blank's work cited above. Theoretical results are compared with experimental data obtained by D. White.

C. J. Bouwkamp (Eindhoven).

Elasticity, Plasticity

Bishop, R. E. D. On dynamical problems of plane stress and plane strain. Quart. J. Mech. Appl. Math. **6**, 250–254 (1953).

In plane stress, the author uses coordinates (x, y, z) chosen so that the stress-free planes coincide with the planes $z=\text{const}$. The equations of motion used are

$$(\lambda'+\mu)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Delta'+\mu\nabla^2(u, v)=\rho\frac{\partial^2}{\partial t^2}(u, v),$$

where $\Delta'=\partial u/\partial x+\partial u/\partial y$, (u, v, w) is the displacement vector, $\lambda'=2\mu\lambda/(\lambda+2\mu)$, and λ and μ are the Lamé constants. These equations are obtained by eliminating w from the equations of motion of classical elasticity. The author's original results are incorrect because he fails to observe that u and v must satisfy conditions of compatibility in order that it be possible to introduce a function w such that

$$\widehat{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0,$$

$$\widehat{yz}=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=0,$$

$$\widehat{zz}=\lambda\Delta'+2\mu\partial w/\partial z=0,$$

and $\partial^2 w/\partial t^2=0$. *J. L. Ericksen* (Washington, D. C.).

Grinberg, G. A. On a method applied by P. F. Papkovič for the solution of the plane problem of the theory of elasticity for a rectangular region and of the problem of bending of a rectangular thin plate with two clamped edges, and on some of its generalizations. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 211–228 (1953). (Russian)

P. F. Papkovič [Stroitel'naya mehanika korablya (Structural mechanics of ships), Gosstrolizdat, Leningrad, 1941] proposed the following problem. To find simultaneous expansions of two independent real functions, defined on the interval $(-\frac{1}{2}b, \frac{1}{2}b)$, in series of the form

$$f_1(y)=\sum a_k L_1[F_k(y)], \quad f_2(y)=\sum a_k L_2[F_k(y)],$$

where a_k are coefficients to be determined, the same in both expansions, $L_i[\cdots]$ are given differential operators with constant coefficients, and $F_k(y)$ are solutions of the differential equation

$$F_k^{(iv)}(y)+2S_k^2F_k''(y)+S_k^4F_k(y)=0$$

with the boundary conditions $F_k(\pm\frac{1}{2}b)$, $F_k'(\pm\frac{1}{2}b)=0$. Even functions of this type are

$$F_k(y)=u_k\sin u_k\cos s_k y-(s_k y)\sin s_k y\cos u_k, \quad u_k=\tfrac{1}{2}bS_k.$$

The greater part of the paper contains an examination in detail of the case in which

$$L_1[F_k(y)]=F_k(y), \quad L_2[F_k(y)]=S_k^2F_k(y)$$

and $f_1(y)$, $f_2(y)$ are even functions. Using an expression given by Papkovič for a_k he sums the series and concludes that the additional condition $f_1(\frac{1}{2}b)=0$ must be imposed. He then applies the method to the case of a plate bounded by arcs of two concentric circles and two radii, clamped along the straight edges and subjected to pressure.

L. M. Milne-Thomson (Greenwich).

Dantu. Méthode nouvelle de détermination des contraintes en élasticité plane. Ann. Ponts Chaussées **122**, 375–405 (1952).

The author considers the analogy between plates and plane elasticity. The method is not new [see, e.g., R. V. Southwell, Quart. J. Mech. Appl. Math. **3**, 257–270 (1950); these Rev. **12**, 372] but the treatment is readily comprehensible to non-mathematicians. Experimental results for a plate are unfortunately compared with those obtained from photo-elasticity which the author states has an accuracy of $\pm 10\%$. It would have been more satisfactory to compare experiment with a simple plane elasticity problem which has a known exact solution. *D. R. Bland.*

Torre, C. Beziehung zwischen den Charakteristiken und einer Berührungstransformation. Österreich. Ing.-Arch. **7**, 32–38 (1953).

The envelope of family of Mohr circles in the stress plane and the yield condition are assumed to be related by a contact transformation. This leads to a relation involving the stresses and their differentials which is expressed in terms of the stresses and the angle between the y-axis and the direction normal to the element on which the stress acts. Two distinguished directions determined by this relation are shown to coincide with the slip line directions. The author relates his work to that of other authors and shows that experimental results are compatible with his basic assumption of the relation between the yield condition and the envelope of the Mohr circles. *N. Coburn.*

Langhaar, H. L. An invariant membrane stress function for shells. J. Appl. Mech. **20**, 178–182 (1953).

The author gropes toward a general solution, expressed in invariant form, of the equilibrium equations $T^{ij}{}_{,j}=0$ on a curved surface of two dimensions. He states that "it seems probable that the general solution . . . is a linear expression containing third or fourth covariant derivatives of F, but the author has not succeeded in finding this generalization." [This problem has been solved by Storchi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 227–331; **8**, 116–120, 326–331 (1950); Rivista Mat. Univ. Parma **3**, 339–360 (1952); these Rev. **11**, 556, 757; **12**, 219; **14**, 1035], who has shown that, in general, fifth derivatives are required and has characterized the surfaces for which fourth, third, or second derivatives suffice.] Apart from some general remarks, the author's attention is directed toward shells of constant curvature, for which he conjectures the solution $T^{ij}=\epsilon^{im}\epsilon^{jn}F_{,mn}+KFg^{ij}$, F being his stress function and K the Gaussian curvature. [This result was similarly conjectured by Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 578–584 (1934), §2]; its generality was proved by Storchi in the second of the notes cited above.] The author discusses in greater detail the case of a shell of revolution, for which he finds the particular form of F. *C. Truesdell* (Bloomington, Ind.).

Serafimov, Peter. Spannungszustand hyperbolischer Schalen nach der Membrantheorie. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 8, 40 pp. (1950). (Macedonian. German summary)

The shell under consideration is a hyperboloid of revolution with upper and lower edges obtained by cutting the shell by two parallel planes normal to the axis. The support is under the lower edge. The author gives the general theory based on the membrane theory and derives the differential equations for arbitrary loads. The equations are solved for the following special cases: (1) a constant distributed vertical load on the whole lateral surface; (2) a constant distributed load on the upper edge; (3) a periodic load distributed on the lateral surface. The stresses are found in all

three cases. The third case constitutes the main part of the work, as the first two cases are quite elementary.

T. Leser (Lexington, Ky.).

Serafimow, Peter R. Beiträge zur Hyperbelschale. I. Hidrostatischer Druck in der Hyperbelschale. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 9, 1–7 (1950). (Macedonian. German summary)

This paper is a continuation of the paper reviewed above. A hyperboloid shell described in this first work is used as a liquid container. The liquid of constant density exerts on the inner surface of the shell a hydrostatic pressure, whose vertical component is a linear function of the depth, other components being zero. The differential equations given in the first work are solved for this case and the stresses are found. A numerical example is appended. *T. Leser.*

Serafimow, Peter R. Beiträge zur Hyperbelschale. II. Verschiebungskomponente für Hyperbelschale bei rotationssymetrischer Belastung. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 9, 9–15 (1950). (Macedonian. German summary)

This paper is a continuation of the paper reviewed second above. A hyperboloid shell under consideration is described in that paper, where the author investigated the stresses only. Referring to his earlier investigations, in this work the author finds the displacements. *T. Leser.*

Hamel, Georg. Über die theorie der dünnen, schwach gebogenen Platten. Z. Angew. Math. Mech. **33**, 138–143 (1953).

The author proposes a self-contained two-dimensional theory of the elastic bending of an isotropic, homogeneous plate with vanishingly small thickness. In this theory, the two twisting moments are not identically equal and the number of boundary conditions along any edge is three. The theory is mathematically self-consistent but nowhere is it established that there exist physical situations to which it may be correctly applied. In the reviewer's opinion it is not justified to reject the present theory solely on the grounds that it differs from the classical Kirchhoff theory. Although this latter theory is generally accepted, the understanding of its physical validity is still far from being as complete as would be liked. The justification of any acceptable theory must rest ultimately upon experiment. *H. G. Hopkins.*

Galimov, K. Z. On the general theory of plates and shells with finite displacements and deformations. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 723–742 (1951). (Russian)

The author continues the researches of two previous papers [Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **1**, 25–46 (1948); **2**, 3–38 (1950); these Rev. **14**, 516] and obtains some new results. By making a hypothesis concerning the distribution of stress through the thickness of the shell, it is possible to set up symmetric tensors for stress and moment which lead to equations of a type similar to those of Castigliano but valid for finite deformations. The author also introduces a variation principle which is applicable to nonlinear problems. [Something on these lines was initiated by N. A. Alumyaè, same journal **14**, 93–98 (1950); these Rev. **12**, 221.]

L. M. Milne-Thomson (Greenwich).

Jung, Hans. Über eine Anwendung der Fouriertransformation in der Plattenstatik. Math. Nachr. **6**, 343–354 (1952).

The author considers the small deflections of a thin rectangular plate under various edge and concentrated loadings. The Fourier transformation is applied to the biharmonic equation satisfied by the unknown deflection, and to the boundary conditions as well. The deflection is then determined as a Fourier integral. The solution is worked out for several examples. In one of these the plate is an infinite strip built-in along one edge, and is subjected to various shears or bending moments on the other edge. In another, the plate is an infinite strip built-in on both edges, and subjected to a general normal load acting in the plane midway between the two edges. Semi-infinite strips and finite plates are also considered. *G. E. Hay.*

Conway, H. D., and Huang, M. K. The bending of uniformly loaded sectorial plates with clamped edges. J. Appl. Mech. **19**, 5–8 (1952).

The deflection of the plate of the title is determined approximately by a superposition method. Two such superpositions are used, one of which seems not less tedious than previously used methods and the other of which would not allow the evaluation of the edge moments. However, the latter allows a ready computation of the deflection.

G. F. Carrier (Cambridge, Mass.).

Deverall, L. I., and Thorne, C. J. Bending of thin ring-sector plates. J. Appl. Mech. **18**, 359–363 (1951).

The deflection of thin elastic plates of ring-sector form is evaluated for a variety of edge conditions and loadings. A product series representation is used. *G. F. Carrier.*

Sen Gupta, A. M. Bending of a cylindrically aeolotropic circular plate with eccentric load. J. Appl. Mech. **19**, 9–12 (1952).

The deflection of a clamped circular anisotropic plate with an eccentrically located concentrated force is found in the form of a product series. *G. F. Carrier.*

Vodicka, V. Durchbiegung von geschichteten Plattenkörpern mit leichter Füllung. Z. Angew. Math. Mech. **33**, 188–199 (1953). (English, French, and Russian summaries)

The author considers the generalization of the usual sandwich plate of three layers to one of $2n+1$ layers, the central and thicker layer being composed of a light weak material. The remaining $2n$ layers are thin, are composed of stronger materials, and are symmetrically disposed with respect to the middle plane both as to thickness and material. Under certain simplifying assumptions the integration of the equations of equilibrium for each layer subject to conditions of continuity at the junctions of the layers leads to the introduction of numerous arbitrary functions. After extensive manipulations the paper ends with a set of partial differential equations whose solution subject to the edge conditions of the plate will enable the determination of these functions. The author states that the procedure at this point will be the same as that used by Prusakov [Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 27–36 (1951); these Rev. **13**, 796] in treating the sandwich plate of three layers. Important papers dealing with the elastic behavior of sandwich plates of three layers are not mentioned.

H. W. March (Madison, Wis.).

Klubin, P. I. The calculation of girder and circular plates on an elastic foundation. Akad. Nauk SSSR. Inženernyĭ Sbornik **12**, 95–135 (1952). (Russian)

The theory of plates on an elastic foundation, as well as tables and practical methods of computation for engineers,

are given by various investigators and the subject, from the point of view of applications, is almost completely covered. All the existing methods, though, require long and complicated computations, and tables are often inaccurate. The author of this work claims that his method is simpler, requires fewer computations, and is sufficiently accurate.

The author begins with plates of infinite length, constant thickness, and constant width (equalling $2l$). The distributed load is independent of the length of the plate, being a function of the width only. The problem of such a plate can be reduced to the problem of a beam on an elastic foundation, whose width equals a unit, and whose length equals $2l$. The determination of vertical displacements involves the well known fourth-order differential equation and the condition that the vertical displacement equals the sag of the foundation. The author's method consists of assuming that the sag of the foundation is an expression containing a series of Chebyshev polynomials. This assumption gives a very simple relation between the sag and the reaction of the foundation, the latter also containing a series of Chebyshev polynomials. By taking no more than four terms of the series the problem is reduced to a system of two simultaneous algebraic equations. The author presents solutions for the following cases of loading: (1) constant loads over the whole plate; (2) constant loads equal but in opposite directions acting on the edges $x=l$, $x=-l$; (3) constant loads in the same direction along the lines $x=a$, $x=-a$.

Next, the author considers plates similar in shape to the first kind, but non-uniform in this respect that the central strip between the lines $x=a$, $x=-a$, has a different moment of inertia from the edge strips. He applies the same method as before but the approximations are not as good as for the first kind of plates. Numerous different cases of loading are presented.

Finally, the author investigates symmetrically loaded circular plates of constant thickness on an elastic foundation. As in the previous cases, the deflection governed by a fourth-order differential equation must equal the sag of the foundation. This time the equation in polar coordinates is not so simple. The author adopts a similar procedure as before, he assumes that the reaction of the foundation and the sag can be expressed as a series of even Legendre polynomials, retains a certain number of terms and reduces the problem to a system of algebraic equations. Like the differential equations, the expressions for vertical displacements are also more complicated, and the author appends tables with numerical values of certain terms in the expressions.

T. Leser (Lexington, Ky.).

Giuliano, Landolino. Sull'equilibrio di una piastra indefinita a forma di striscia, incastrata e appoggiata. Ann. Scuola Norm. Super. Pisa (3) **6** (1952), 147–171 (1953).

In un lavoro pubblicato alcuni anni addietro Ghizzetti [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 145–187 (1947); questi Rev. **9**, 121], supponendo soddisfatte dalle funzioni $f(x, y)$ e $u(x, y)$ le condizioni sufficenti per rendere possibile passare, mediante una transformazione di Fourier rispetto alla variabile y, dall'equazione $\Delta\Delta u(x, y)=f(x, y)$ all'associata

$$v^{(4)}-2\lambda^2v''+\lambda^4v=f^*(x, \lambda),$$

$$f^*=\int_{-\infty}^{\infty}e^{-\lambda y}f(x, y)dy,\ v=\int_{-\infty}^{\infty}e^{-\lambda y}u(x, y)dy,$$

ha studiato dettagliatamente il problema della determinazione, per questo mezzo, di una funzione di Green atta a rappresentare la $u(x, y)$ come integrale di $f(x, y)$ per il caso di una piastra incastrata lungo due lati rettilinei paralleli, di lunghezza infinita. Nella presente memoria, l'A., seguendo da vicino il procedimento di Ghizzetti, studia il caso della piastra incastrata lungo un lato e semplicemente appoggiata lungo l'altro lato, giungendo a conclusioni in tutto analoghe alle precedenti. *B. Levi* (Rosario).

Lur'e, A. I. Stressed state about an ellipsoidal cavity. Doklady Akad. Nauk SSSR (N.S.) **87**, 709–710 (1952). (Russian)

The problem of the stress distribution around an ellipsoidal cavity in an infinite elastic medium is reconsidered. This problem was solved previously by M. A. Sadowsky and the reviewer [J. Appl. Mech. **16**, 149–157 (1949); these Rev. **10**, 760]. In the present paper the generating Boussinesq-Papkovich stress functions are assumed as improper incomplete elliptic integrals of the first and second kind, and the use of Jacobian elliptic functions, which is necessary to effect a reduction of the solution to tabulated functions, is avoided. A gain in transparency is claimed.

E. Sternberg (Chicago, Ill.).

Šeremet'ev, M. P. Elastic equilibrium of an elliptic ring. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 107–113 (1953). (Russian)

The author investigates a ring bounded by two confocal ellipses, the loads being applied on the boundaries. A. Timpe [Math. Z. **17**, 189–205 (1923)] solved the above problem, but Mushelišvili [Some fundamental problems of the mathematical theory of elasticity, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949, p. 231; these Rev. **11**, 626] pointed out that Timpe's solution is incorrect. The author gives now a correct solution. He uses a conformal mapping method presented in an earlier work [Ukrain. Mat. Žurnal **1**, no. 3, 68–80 (1949); these Rev. **13**, 886]. A numerical example of a ring with given confocal ellipses as boundaries, and the outer boundary compressed by a constant distributed normal load illustrates the use of general formulas.

T. Leser (Lexington, Ky.).

Buharinov, G. N. On the problem of equilibrium of an elastic circular cylinder. Vestnik Leningrad. Univ. **1952**, no. 2, 3–23 (1952). (Russian)

A uniform circular cylinder of length $2h$ is acted upon by uniform pressure over one end only. The lateral surface is constrained so that on it the radial displacement u is zero and the shearing force is $a+bz^2$, where the z-axis is along the axis of the cylinder. Assuming infinitesimal elastic deformation, the author calculates the radial displacement u at distance r from the axis and the longitudinal displacement w in the form of infinite series involving the Bessel functions $J_0(x)$, $J_1(x)$, where $x=\lambda_k r/h$ and the hyperbolic functions $\sinh y$ and $\cosh y$, where $y=\lambda_k z/h$, λ_k ($k=1, 2, 3, \cdots$) being the zeros of $J_1(x)$.

L. M. Milne-Thomson (Greenwich).

Lesky, Peter. Determinazione degli stati di tensione piana in un cilindro elastico a sezioni ellittiche. Ann. Scuola Norm. Super. Pisa (3) **6** (1952), 255–267 (1953).

Strictly plane stress systems, in which the stress tensor is plane, have been studied by A. Ghizzetti [Ann. Mat. Pura Appl. (4) **29**, 125–130 (1949); these Rev. **12**, 63]. The author exhibits the solution, expressed in terms of the

Green's function and other infinite series of elementary functions, for an elliptic cylinder on whose boundary surface the normal component of stress is prescribed arbitrarily.

C. A. Truesdell (Bloomington, Ind.).

Bolotin, V. V. Integral equations of constrained torsion and of stability of thin bars. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 245–248 (1953). (Russian)

The equations

$$EI_y u^{(IV)} + (M\theta)'' = 0, \quad EI_\omega \theta^{(IV)} - GI_d \theta'' + Mu'' = 0,$$

give $u(z)$, the lateral deflection of a thin bar, and $\theta(z)$, the angle of rotation of a section, under action of a bending moment $M(z)$. The author writes $B = -EI_\omega \theta''$ and combines to get $B'' - KB = -M^2\theta/EI_y$. He then chooses a Green's function, converting the system to an integral equation, and proposes a method of finding critical loads through use of successive approximations and approximate integrations.

R. E. Gaskell (Seattle, Wash.).

***Southwell, R. V., and Vaisey, Gillian. A problem suggested by Saint-Venant's 'Mémoire sur la torsion des prismes'.** Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, pp. 100–106. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953.

This paper uses relaxation methods to solve the torsion problem for a bar of rectangular cross-section. While the analytical solution has been given by Saint-Venant, it is shown that the labor of computing a numerical solution is less by relaxation than by use of the analytical solution. The authors present diagrams for a series of rectangles with different ratios of length to breadth showing in an interesting manner how the solution passes from the case of two lines of symmetry for elongated rectangles to four lines of symmetry for rectangles nearly (or actually) square. The critical ratio of length to breadth is given by Saint-Venant as 1.4513.

W. E. Milne (Corvallis, Ore.).

Rostovcev, N. A. On the solution of a plane contact problem. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 99–106 (1953). (Russian)

This publication is based on two works of I. Ya. Štaerman (Steuermann) [C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 179–181 (1940); The contact problem of the theory of elasticity, Gostehizdat, Moscow-Leningrad, 1949; these Rev. **2**, 332; **11**, 756]. The problem can be stated as follows: the surfaces of two bodies before deformation have one point of contact; after being compressed by constant forces, P, the surfaces contact along a segment $-a \leq x \leq a$ (or throughout an area of a circle $r \leq a$). Find the pressure between the contacting segments (or surfaces). The distance between the segments (or surfaces) before the deformation is assumed to be $z_1(x)+z_2(x)$ (or $z_1(r)+z_2(r)$), where z_1 and z_2 are arbitrary functions with usual conditions of continuity and differentiability. (In Hertz theory z_1+z_2 is a quadratic polynomial.) Under these assumptions the pressure $p(x)$ can be expressed as an improper integral shown by Štaerman [loc. cit.]. The author of this paper demonstrates that this improper integral can be transformed into a Volterra integral equation. After the presentation of the general theory the author investigates the following special cases: (1) $z_1(x)+z_2(x)=A|x|^n$ $(n>1)$, and examines in detail the position of the maximum pressure for different values of n; (2) one surface is a plane, another is a wedge; (3) one surface is a plane, another is a base of a rigid circular cylinder (termed punch) with rounded edges.

T. Leser (Lexington, Ky.).

Leonov, M. Ya. The general problem of the pressure of a circular punch on an elastic half-space. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 87–98 (1953). (Russian)

The theory of the pressure of a rigid circular cylinder (termed punch) on the elastic half-space was given in a most general form by L. A. Galin [same journal **10**, 425–448 (1946); these Rev. **8**, 241]. The author of this paper presents a summary of the above work and gives a more detailed development for particular cases. The method involves a wide application of harmonic stress functions, potential functions, and Green functions. The problem can be stated briefly as follows: given the displacements under a rigid circular punch, find the pressure under the punch and the displacements of the plane surface of the half-space which is outside the punch. The author discusses two cases: when the axis of the cylindrical punch is normal to the plane surface of the half-space, and when it is inclined.

T. Leser (Lexington, Ky.).

Wuest, W. Der Einfluss der Querschnittsform auf das Verhalten von Bourdonfedern. Ing.-Arch. **20**, 116–125 (1952).

The author presents an approximate theory for stresses and deformations of Bourdon pressure gages. The problem belongs in the field of thin toroidal elastic shells. The author's approach, which makes possible very considerable simplifications compared to earlier treatments, amounts to treating the toroidal shell as composed of two nearly cylindrical shells. Strips of each of these two shells behave like beams on an elastic foundation. The author compares some of his results with earlier work on toroidal shells with elliptical cross section [Clark, Gilroy and E. Reissner, J. Appl. Mech. **19**, 37–48 (1952); these Rev. **14**, 429] finding appreciable numerical differences. The reviewer believes that the cause for these discrepancies has to do with the boundary conditions assumed by the author, the range of validity of these boundary conditions possibly being not as wide as indicated by the author.

E. Reissner.

Behlendorff, Erika. Über die Bestimmung der Wärmespannungen in einer Kugel. Math. Nachr. **8**, 59–64 (1952).

By various ingenious transformations, the author obtains a solution in closed form (that is, in terms of an infinite integral) of the following problem. Determine the deformations of a solid elastic sphere, the material of which obeys Hooke's Law when the surface of the sphere is free of stress and the temperature over the surface is prescribed. The author refers to a less simple solution of her problem, not in closed form, by O. Tedone [Rend. Circ. Mat. Palermo **17**, 241–274 (1903)].

E. Reissner (Cambridge, Mass.).

Buckens, F. Détermination des tensions thermo-élastiques dans un tube cylindrique. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **27**, no. 5, 44 pp. (1952).

The author is primarily concerned with determining thermo-elastic stresses in a thick walled cylindrical tube. The temperature T and displacement vector D are assumed to satisfy the equations

$$\Delta T = 0,$$
$$(\lambda+2\mu)\ \text{grad div}\ D - \mu\ \text{curl curl}\ D - \text{grad}\ \varphi = 0,$$

where Δ is the Laplacian operator, λ and μ are the Lamé constants, $\varphi = \alpha(3\lambda+2\mu)T$, where α is the coefficient of thermal expansion. The cylindrical surfaces of the tube are assumed to be force free, the plane ends subject to given

surface tractions. In cylindrical coordinates (r, θ, z), the temperature is assumed to satisfy boundary conditions of the form $a\partial T/\partial r+\alpha T=f(\theta, z)$ on the cylindrical surfaces, $b\partial T/\partial z+\beta T=g(r, \theta)$ on the plane ends, where a, α, b, and β are constants. The author describes a method for obtaining solutions of the basic equations satisfying all boundary conditions exactly except the stress conditions on the plane ends. On the plane ends these solutions can be made to yield correct resultant forces and moments. The method used is to expand the displacement components and T in double Fourier series in θ, z with coefficients which are functions of r and to evaluate these coefficients so as to satisfy the conditions described above. *J. L. Ericksen.*

Kammerer, Albert. Stabilité de l'équilibre élastique des matériaux. Cas des corps sans cohésion et des liquides. C. R. Acad. Sci. Paris **234**, 1735–1736 (1952).

In an earlier paper [same C. R. **231**, 681–683 (1950); these Rev. **12**, 219] the author considered the equilibrium of a body whose principal stresses σ_i are quadratic functions of the principal strains e_i, in terms of the geometry of the σ_i space. In the present note, the author extends these notions to bodies for which Young's modulus vanishes. Included here is the incompressible liquid, for which Poisson's ratio is equal to one half. *G. E. Hay.*

Gray, C. A. M. The analysis of infinitely long beams under normal loads. J. Proc. Roy. Soc. New South Wales **85** (1951), 20–25 (1952).

The author considers the plane strain of a body whose cross-section is an infinite strip, with loading normal to the edges of the strip. The method of Muskheliŝvili is used, so that the problem is reduced to the determination of two analytic functions of a complex variable. The cross-section of the body is mapped conformally onto the unit circle, and the two required functions are expressed as power series with coefficients found in the usual way to satisfy an infinite set of equations. These equations are solved by iteration. The special case when the loading consists of two equal and opposite forces, one acting on each edge, is worked out in detail. *G. E. Hay* (Ann Arbor, Mich.).

Agostinelli, Cataldo. Sulla risoluzione mediante integrali definiti del problema delle vibrazioni di una piastra. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 396–400 (1952).

Let σ denote a closed region in the (x, y)-plane, and let D denote the set of points (x, y, t) for which $t \geqq 0$ and (x, y) belong to σ. In an earlier paper [Ann. Mat. Pura. Appl. (4) **26**, 27–41 (1947); these Rev. **10**, 124] the author solved the following boundary value problem for σ restricted to one of two shapes, namely rectangular or circular. The unknown function $u(x, y, t)$ is in D to satisfy the differential equation $\Delta\Delta u+\alpha^2\partial^2 u/\partial t^2=0$, in σ $u(x, y, 0)=\varphi(x, y)$, $u_t(x, y, 0)=\theta(x, y)$, and on the boundary of σ the function u and its normal derivative are to vanish. Here $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, α is a constant, and the assigned functions φ and θ are assumed to have derivatives of all orders. In the present paper the Laplace transform is used to treat this problem without the above mentioned restrictions on the shape of σ. *F. G. Dressel* (Durham, N. C.).

Ericksen, J. L. On the propagation of waves in isotropic incompressible perfectly elastic materials. J. Rational Mech. Anal. **2**, 329–337 (1953).

The equations of motion, continuity, and the stress-strain relations of a homogeneous, isotropic, incompressible, perfectly elastic material in the form due to Truesdell [same J. **1**, 125–171, 173–300 (1952); these Rev. **13**, 794] are considered. By use of Hadamard's method [Leçons sur la propagation des ondes . . . , Hermann, Paris, 1903], the author expresses the discontinuities in the derivatives of pressure, the displacement gradients, and the velocity in terms of Lagrangian variables. These relations are transformed into Eulerian variables, and it is shown that the incompressibility condition implies that the discontinuity front is always a transverse wave. From the equations of motion, the author obtains an expression for the velocity of propagation of the wave and shows that a vector in the direction of the discontinuity front is the eigenvector of a system of linear homogeneous equations. When the strain energy is of the form used in the Mooney-Rivlin theory of rubber, then the above relations furnish real values for the velocity of propagation. Finally, it is indicated that the general theory of higher-order waves is similar to that of the above second-order waves. *N. Coburn.*

Kusukawa, Ken-ichi. On the shock wave in the elastic medium. J. Phys. Soc. Japan **7**, 200–203 (1952).

The shock wave in an elastic medium as generated by the application of particle velocities greater than the acoustic speed of the medium is discussed. The reflection of such waves from a rigid wall, under the assumption that the medium exhibits the same elastic properties on each side of the shock, is calculated. *G. F. Carrier.*

Satô, Yasuo. Velocity of elastic waves propagated in media with small holes. Bull. Earthquake Res. Inst. Tokyo **30**, 179–190 (1952). (Japanese summary)

In order to account for porosity of material, the elastic constants and the velocity of elastic waves are expressed in terms of relative density $\rho=\delta/\delta_0$, where the density of the real material is δ_0, and δ that of an actual material. To compute the velocities of P- and S-waves, the author makes use of expressions for elastic constants in case of empty holes obtained by J. K. Mackenzie [Proc. Phys. Soc. Sect. B. **63**, 2–11 (1950)], and extends the results to the case of spherical holes filled with liquid. *W. S. Jardetzky.*

Satô, Yasuo. Study on surface waves. VIII. Nomogram for the phase velocity of Love-waves and maximum thickness of the surface layer. Bull. Earthquake Res. Inst. Tokyo **31**, 81–87 (2 inserts) (1953). (Japanese summary)

***Thomson, William T. Transmission of pressure waves in liquid filled tubes.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 927–933. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author studies the propagation of pressure waves in thin elastic tubes filled with a compressible inviscid fluid, taking into account the shear forces in the tube wall which were neglected by previous investigators. He finds that, corresponding to any given frequency, there are a finite number of modes, one of which has a phase velocity smaller than the velocity of sound in the fluid, while all others have a velocity larger than that of sound in the fluid. An estimate is made of the attenuation due to the viscosity of the fluid. Numerical computations are presented for a typical case.

The reviewer has some doubt concerning the validity of the author's equations for the motion of the tube for all frequencies. The expressions for the longitudinal and hoop stresses in terms of strains are approximate and the terms

which are neglected [see Love, Mathematical theory of elasticity, 4th ed., Cambridge, 1934, §334] give rise to terms in the equations of motion which are important when compared with the terms arising from the shear force whenever the wave length of the disturbance is comparable with or larger than the radius of the tube. *G. W. Morgan.*

***Koiter, W. T. On partially plastic thick-walled tubes.** Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, pp. 232–251. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953.

Analysis is given of the plastic deformation, under internal pressure, of long uniform tubes made of a perfectly-plastic material that obeys the yield condition of Tresca and the associated stress versus strain-rate equations. Solutions applicable to all but very thick-walled tubes are found for the problems of zero axial strain, 'open' ends, and 'closed' ends. The modification of these solutions to include very thick-walled tubes is discussed, and the complete solution is given for the problem of zero axial strain. Attention is directed to the need for further analysis that takes into account finite plastic deformations in such cases. Comparison of numerical results with those few available numerical results derived from an analysis based upon the yield condition of von Mises [Hodge and White, J. Appl. Mech. **17**, 180–184 (1950); these Rev. **12**, 562] shows good agreement (if the yield limits in shear are the same for both yield conditions) except for axial stress and strain. In comparison with previous analysis [e.g., Hodge and White, loc. cit.] the present approach leads to an extremely simple solution in closed form; moreover this solution, so far as application to engineering design is concerned, is quite satisfactory over a wide range of cases of interest. *H. G. Hopkins.*

Kempner, Joseph, and Pohle, Frederick V. On the non-existence of a finite critical time for linear viscoelastic columns. J. Aeronaut. Sci. **20**, 572–573 (1953).

Inoue, Nobuo. Application of the theory of supersonic flow to the two-dimensional isostatical problem in the theory of plasticity. J. Phys. Soc. Japan **6**, 460–465 (1951).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Thomescheit, A. Die Lichtverteilung in geometrisch-optischen Zerstreuungsfiguren für beliebige Einstellebenen bei optischen Systemen aus sphärischen Flächen. Optik **10**, 221–236 (1953).

A calculation procedure is derived for obtaining the geometric-optical light distribution in an arbitrary plane for the case of a meridional bundle of rays. A comparison is made with the appropriate results of Staeble, Malinowski, and Huber. The case of skew bundles is then treated, and consideration is given to actual computation with the derived equations. The calculation of the focal point of an elementary bundle is described and also the influence of reflection and absorption losses on the light distribution.

E. W. Marchand (Rochester, N. Y.).

Ishiguro, Kôzô, and Katô, Toshio. The reflection and transmission of a multi-layer film. J. Phys. Soc. Japan **8**, 77–81 (1953).

The authors introduce amplitude reflection and transmission coefficients (both for the light incident from one and from the other side) in Weinstein's matrix formula connecting the amplitudes of the electric vector of light in two different homogeneous layers of a multi-layer film. From the corresponding matrix equation formulas for transmissivity and reflectivity of an arbitrary multi-layer film are derived; the main results are the same as the results of Abeles, derived without use of matrix formulation. Some special cases are discussed; further applications will be given in another paper. *M. Brdička* (Prague).

Bouwkamp, C. J. Diffraction theory. A critique of some recent developments. With an additional section, Diffraction by an aperture in a planar screen, by N. Marcuvitz. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-50**, iii+91 pp. (1953).

This report is the product of a semester which the author recently spent as a member of the Washington Square College Mathematics Research Group. During that time he surveyed and assessed the literature on the theory of diffraction, and the present report, on the subject of diffraction by apertures, does not merely relate the efforts of many workers, but also, where necessary, criticises, emends, and pursues the consequences of their work. The result is an extremely valuable critique of the whole subject.

The topics discussed include recent modifications of Kirchhoff's theory, the theory of small apertures, Babinet's principle for plane obstacles, variational methods, and singularities at sharp edges. The exact solutions recently obtained for diffraction by circular apertures and disks are not discussed, nor are the applications of the Wiener-Hopf technique.

For completeness, a discussion from an alternative viewpoint of the problem of diffraction by an aperture, written by Professor N. Marcuvitz, has been included in the report. This alternative method emphasizes the vector aspect of the diffraction problem. A dyadic Green's function formalism is used to obtain a transverse vector integro-differential equation for the transverse electric field in the aperture. The knowledge of the aperture field permits the direct calculation of the electromagnetic field at any point. The method is illustrated by a discussion of the diffraction of plane waves by a circular aperture. *E. T. Copson.*

Sollfrey, William. Diffraction of pulses by conducting wedges and cones. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-45**, i+15 pp. (1952).

Diffraction of a plane pulse by a conducting wedge or cone is obtained by first finding the solution due to a plane wave [H. M. Macdonald, Electromagnetism, Bell, London, 1934, pp. 79, 144] and then integrating over the propagation constant of this wave. For the wedge the solution agrees that that previously obtained by Keller and Blank [Comm. Pure Appl. Math. **4**, 75–94 (1951); these Rev. **13**, 304]. The analysis for the cone problem involves contour integration and other transformations containing spherical Bessel functions and Legendre functions of fractional degree. Only the case of normal incidence is worked out.

C. J. Bouwkamp (Eindhoven).

Twersky, Vic. Multiple scattering of radiation. II. The grating. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-39**, iii+50+ii pp. (1951).

Verf. behandelt die Beugung einer normal einfallenden ebenen Welle an einem endlichen Gitter von parallelen geraden Kreiszylindern unendlicher Länge. Hierbei wird die mehrfache Beugung und Zerstreuung der Wellen durch die einzelnen Zylinder in Betracht gezogen. Zunächst leitet Verf. für den Fall einer einfallenden akustischen Welle einem allgemeinen Ausdruck für das Geschwindigkeitspotential der Biegungswellen ab, in Form unendlicher Fourier-Hankel-Reihen. Diese allgemeine Formel wird angewandt auf den Fall, dass der Zylindenabstand grösser ist als die Wellenlänge. Sodann werden die Endeffekte vernachlässigt und wird eine explizite Lösung in Reihenform angegeben. Aus dieser Lösung gelangt Verf. durch systematische Vernachlässigungen zu Ausdrücken, welche numerische Ergebnisse für verschiedene Fälle abzuleiten erlauben. Diese werden kurvenmässig ausführlich diskutiert. Sodann wendet Verf. sich der Aufgabe zu, dass das obengenannte Gitter ersetzt wird durch eine Reihe von parallelen Halbkreiszylindern, welche auf einer unendlichen vollkommen leitenden Ebene aufgesetzt sind. Auch für diesen Fall gelangt er zu numerischen Ergebnissen, welche ausführlich kurvenmässig diskutiert werden. Zum Schluss wird die Theorie ausgedehnt auf Gitter mit anderen Elementen als Zylinder. *M. J. O. Strutt* (Zürich).

Ament, W. S. Toward a theory of reflection by a rough surface. Proc. I. R. E. **41**, 142–146 (1953).

The author approaches the problem of a plane electromagnetic wave impinging on a perfectly reflecting plane with statistically specified corrugations, the set-up being effectively two-dimensional, with the aim of estimating the reflection coefficient. An integral equation of the first kind is found for a surface current-density, and various averaged relations are set up, with the aid of which a suggested reflection coefficient (attributed to Pekeris and MacFarlane) is discussed. Calculation of the effect of an autocorrelation function in any standard form has so far been prevented by analytic difficulties, so that the author's formulae have an interim character. *F. V. Atkinson* (Ibadan).

Nadile, Antonio. Sulla propagazione delle onde elettromagnetiche entro un cavo cilindrico riempito di dielettrico eterogeneo. Matematiche, Catania **7**, 1–17 (1952).

The problem considered in this paper is the propagation of electromagnetic waves inside a tube bounded by perfectly conducting circular cylinders whose equations are $r=a$ and $r=b$ in cylindrical coordinates (r, θ, z). The medium in the tube is assumed to have constant magnetic permeability, but the dielectric constant is proportional to $1/r^2$. The author considers only harmonic waves in which: (i) the magnetic vector is perpendicular to the axis of the tube; (ii) the field is independent of the angle variable θ, so that

$$\mathbf{E}=\mathfrak{E}(r)e^{i(\omega t-\beta z)}, \quad \mathbf{H}=\mathfrak{H}(r)e^{i(\omega t-\beta z)}.$$

The solution depends on solving the differential equation

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(\frac{\alpha^2}{x^2}-1\right)y=0$$

of confluent hypergeometric type. This equation is solved in two ways: (i) by power series, (ii) by a complex Laplace transform. The principal result is the determination of a lower bound to the frequency if the resulting motion is not to be evanescent. *E. T. Copson* (St. Andrews).

Kales, M. L. Modes in wave guides containing ferrites. J. Appl. Phys. **24**, 604–608 (1953).

When a static magnetic field is applied in the longitudinal (z-) direction to a wave guide containing a ferrite material, the medium in the guide becomes anisotropic. Polder [Philos. Mag. (7) **40**, 99–115 (1949)] has shown that when a micro-wave field is propagating in such a medium the relations between the components of magnetic induction and field strength are given by $B_x=\mu H_x-j\mu' H_y$, $B_y=\mu H_y+j\mu' H_x$, $B_z=\mu_z H_z$, where the μ's depend on the static magnetic field. Using Polder's equations, the author investigates the possible modes within a wave guide of arbitrary cross-section; all components are assumed of the form

$$f(x, y)\exp(\gamma z+j\omega t).$$

He proves that there are no TE, TM or TEM modes. The special case of a circular guide is examined in detail. A transcendental characteristic equation is derived for the propagation constant γ in terms of the Bessel function J_n corresponding to an angular dependence $\exp(jn\theta)$ of the transverse field, when the guide is wholly or partly filled with the ferrite. *C. J. Bouwkamp* (Eindhoven).

Clemmow, P. C. Radio propagation over a flat earth across a boundary separating two different media. Philos. Trans. Roy. Soc. London Ser. A. **246**, 1–55 (1953).

In an attempt to explain certain phenomena observed in terrestrial radio wave propagation, such as the recovery-effect in the case of mixed-path attenuation, the author presents a detailed mathematical analysis of two different physical models. In both models the earth is considered flat, and the antenna, emitting vertically polarized waves, is represented by a line source parallel to the earth, so that the analysis becomes two-dimensional. In the first model the earth is assumed homogeneous and of finite permittivity except for a sheet at the surface in the form of a perfectly conducting half-plane. This problem is rigorously solved by means of the Wiener-Hopf technique of the theory of integral equations. In the second model, the earth consists of two different media separated by a plane interface perpendicular to the earth's surface. Unlike the first, the second problem cannot be solved in a rigorous way. An approximate solution is derived on the assumption of simplified boundary conditions at the surface of the earth [cf. Feinberg, Acad. Sci. USSR. J. Phys. **10**, 410–418 (1946); these Rev. **8**, 300]. The author discusses in detail the relation of his models to actual cases met in practice. Two numerical examples show how close Millington's [Proc. Inst. Elec. Engrs. Part III. **96**, 53–64 (1949)] "engineering" solution is to the author's. *C. J. Bouwkamp* (Eindhoven).

***Toraldo di Francia, Giuliano. Onde elettromagnetiche.** Nicola Zanichelli Editore, Bologna, 1953. xiii+286 pp. 3000 lire.

This book is based on courses of lectures which the author has given for several years in the Faculty of Physics of the University of Florence. The scope of the work is very wide and is, as a result, very superficial.

After a 41-page mathematical introduction in which vectors, tensors, Fourier series and integrals, and the principle of stationary phase are discussed, there follow thirteen chapters which deal with the following topics: Giorgi's system of units, Maxwell's equations, propagation of an electromagnetic field, the electromagnetic potentials,

special relativity, electric circuits and transmission lines, plane waves, waves in general, reflection and refraction, diffraction, guided waves, the wave guide as a transmission line, resonant cavities. All this is presented in an interesting way. But there are no references to the literature of the subject—even the occasional historical notes make no reference to any workers in the 20th century except Einstein and Minkowski. This seems to be a poor way of presenting a subject which is very much alive. The bibliography gives only a list of ten standard reference books.

E. T. Copson (St. Andrews).

Volkov, D. M. A second integral giving a new conservation law for electromagnetic fields. Vestnik Leningrad. Univ. 1952, no. 2, 42–43 (1952). (Russian)

For a field active inside an empty finite region D, bounded by a metallic surface S, the author gives the time-constancy of

$$\mathbf{E}_2=\int\int\int_D \sum_{k,l=1}^{3}(|H_l^k|^2+|E_l^k|^2)dx_1dx_2dx_3 - \int\int_S\left(|E|^2\sum_{j=1}^{2}P_{ij}+\sum_{j=1}^{2}{}^*H^j\sum_{k=1}^{2}P_{kj}\,{}^*H^k\right)dS.$$

Here, in the space-integral, H_l^k, E_l^k are the space-derivatives of the field components; in the surface-integral the x_r^* are local coordinates with x_3^* the normal, $x_3^*=P(x_1^*, x_2^*)$ the local equation of S, and ${}^*H^k$ the corresponding field components. A proof is sketched. $\mathbf{E}_2$ is stated to be independent of the energy, and to measure the excitation of the field. There are said to be other such results involving higher derivatives.

F. V. Atkinson (Ibadan).

Šehter, V. On a system of positive integrals of Maxwell's equations. Doklady Akad. Nauk SSSR (N.S.) **89**, 619–622 (1953). (Russian)

For an electromagnetic field in an empty finite region Ω bounded by a perfectly conducting surface S the author considers sets of positive quadratic functionals which are constant in time. For integral $k\geqq 2$ a set is termed complete if the boundedness of the functionals implies the boundedness in mean square of the spatial partial derivatives of the field of orders from zero to k. It is shown, with differentiability restrictions on S, that the set

$$\int_\Omega[(\partial^m\mathbf{E}/\partial t^m)^2+(\partial^m\mathbf{H}/\partial t^m)^2]d\Omega,\quad m=0,\cdots,k,$$

is complete. A more special result had been given by D. M. Volkov [see the paper reviewed above].

F. V. Atkinson (Ibadan).

De Sloovere, H. La stabilité, au sens de Th. De Donder, des lois électromagnétiques de Maxwell. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 183–185 (1953).

The author investigates the second variation of an integral which is stationary for a Maxwellian field, and obtains an integrand with mixed signs. But the positive terms are multiplied by the small factor c^{-2}, and so he regards the integrand as negative definite and concludes that the Maxwellian laws are stable, in the sense of Th. De Donder.

J. L. Synge (Dublin).

Hampel, R. Quelques applications des équations intégrales dans la théorie d'électricité. Prace Mat.-Fiz. **48**, 79–100 (1952).

The two problems discussed relate to an electrostatic field perturbed by a body of general shape and varying dielectric constant, and to a stationary current distribution in a body with varying conductivity. It is shown in each case that the solution can be reduced to a double application of the Fredholm theory. The first stage is to find a simple layer distribution over the surface of the body giving a harmonic Green's function which, while not the Green's function of the actual problem, can serve as the kernel in an integral equation which gives the desired solution. The existence of this Green's function appears completely established with the additional restriction that the varying parameter be constant on the surface of the body. Full attention is paid throughout to rigour.

F. V. Atkinson (Ibadan).

Shen, D. W. C. Synthesis of uniform transmission line. Electronic Engrg. **25**, 287–289 (1953).

The uniform transmission line is approximated by dividing the line into n identical sections constructed, for example, in a T, Π, or equivalent form with lumped elements. Let $[M]$ be the transmission matrix of an elementary section of the lumped circuit approximation. A mathematical proof is adduced to show that the limiting value of $[M]^n$ as n approaches infinity is equal to the transmission matrix of the uniform line, the latter having been derived from independent considerations.

R. Kahal (Monterey, Calif.).

Gaponov, A. V. On a dynamical model of the general theory of electrical machines. Doklady Akad. Nauk SSSR (N.S.) **89**, 45–48 (1953). (Russian)

This sketchy paper applies the ideas of another one [same Doklady (N.S.) **87**, 401–404 (1952); these Rev. **14**, 825] to the "concrete" problem of an idealized, completely linear, electrical "parent" machine whose stator and rotor are represented by two infinitely long and infinitely thin coaxial cylindrical shells with distributed surface currents parallel to their common axis (these currents represent those in the winding). The currents are assumed to be developable into Fourier series in the azimuths θ and θ' on the stator and the rotor, both measured from axial planes fixed in the respective cylinders. The coefficients of these series are regarded as generalized velocities of a Lagrangian system with a countable number of freedoms (the Lagrangian is the sum of the kinetic and magnetic energy). The validity of the (infinitely many) Lagrange equations is assumed. If the rotor current is expressed in terms of the stator angle θ, the coefficients of the expansion are recognized as nonholonomic general velocities, and the validity of the Boltzmann-Hamel equations is claimed for them. The author elects to regard the machine with a finite number of coils in the winding as the outcome of imposing infinitely many constraints on the "parent" machine, with a finite degree of freedom resulting. These constraints are nonholonomic if there are commutators. One gets the impression that the author considers postulating the validity of the equations for the infinitely many parameters of the "parent" machine logically preferable to doing this for the finite number of parameters of the "constrained" machine. This seems the more strange that the parameters of the "parent" machine are not even "normal" (separated), as is the case in the Rayleigh method for vibrations of continuous systems. Another strange remark is that G. Kron's "primitive" network (a finite number of disconnected coils) is not acceptable as the "parent" machine because it lacks "concrete physical meaning".

A. W. Wundheiler (Chicago, Ill.).